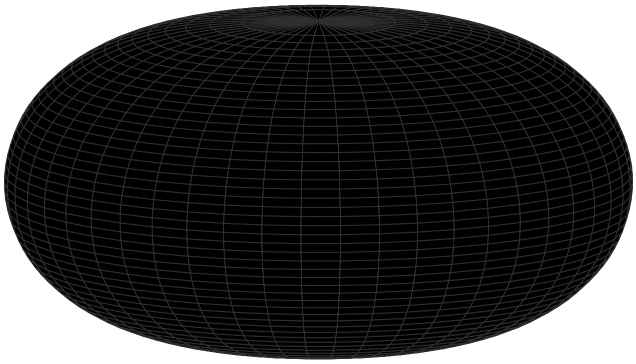
Black Holes for First-Time Physicists



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Abstract: Black holes are some of the most fascinating objects in the universe. They are regions of spacetime in which gravity is so intense nothing can escape them. Since they constitute some of the most extreme scenarios of general relativity, it is curious that they can be described by only a few numbers such as mass, charge, and angular momentum. This book discusses the basic ideas about black hole mechanics and how to make simple computations about complex objects. A few applications we will consider are how much energy can gravitational waves carry away in a black hole merger, how much charge can astrophysical black holes have, whether it is possible to turn a black hole into a naked singularity by overcharging or over-spinning it, how to extract energy from black holes, and much more.

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Cover: geometrically accurate depiction of the event horizon of a rapidly rotating Kerr black hole ($J = \frac{\sqrt{3}G}{2c} M^2$) on three-dimensional Euclidean space. If the black hole was spinning faster, the horizon geometry would be so curved it would not be possible to represent it in three-dimensional Euclidean space [1].

Contents

Preface

> Of all the entities I have encountered in my life in physics, none approaches the black hole in fascination. And none, I think, is a more important constituent of this universe we call home. The black hole epitomizes the revolution wrought by general relativity. It pushes to an extreme—and therefore tests to the limit—the features of general relativity (the dynamics of curved spacetime) that set it apart from special relativity (the physics of static, "flat" spacetime) and the earlier mechanics of Newton. Spacetime curvature. Geometry as part of physics. Gravitational radiation. All of these things become, with black holes, not tiny corrections to older physics, but the essence of newer physics.
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> John Archibald Wheeler [2, p. 312]

This book started in 2024, when I decided to apply as a tutor to the V ICTP-SAIFR Summer School for Young Physicists. This is an innovative summer school which allows high school students to experience a little bit of advanced topics in physics. The projects developed by the students can involve pretty advanced physics, but the idea is still that they get their hands on the numbers. While thinking about which topic I could cover on a project for the summer school, I thought about black holes.

Black holes are some of the most mysterious regions of spacetime. They are a prediction of general relativity—presently our best theory for describing all gravitational phenomena—that challenge a lot of common sense. These regions of spacetime involve such intense gravitational fields that they literally trap everything that enters them, and not even light can escape their influence.

The full description of black holes involves intricate mathematical calculations relying on differential geometry. Differential geometry is a generalization of Euclidean (or plane) geometry to situations in which space can be curved. This means, for example, that parallel lines may cross, the internal angles of triangles may add to more or less than 180°, and so on. When these ideas are applied to space and time simultaneously while following the laws of relativity, we obtain a mathematical theory describing the gravitational aspects of our universe to astonishing precision.

Differential geometry is a difficult subject, and there are profound monographs discussing many different aspects of black hole physics. It may come as a surprise, then, that some calculations in black hole physics can be carried out with elementary algebra. I noticed this when I recalled one of the problems on chapter 12 of the famous general relativity textbook by Wald [3], which has a reputation for being fairly advanced. It was problem 4, discussing the energy emitted in a collision of two rotating black holes. While this particular problem was based on actual physics papers [4–6], it surprisingly could be solved with simple methods once you knew the right tricks. Thinking about it made me wonder how many other problems in black hole physics could be addressed in a similar way, with high school methods at most. It turned out to be more than you would expect.

You see, black holes in equilibrium are characterized by a handful of quantities—typically the mass, electric charge, and spin—which obey a few simple laws, nowadays known as the laws of black hole thermodynamics. While these laws are actually detailed differential geometry theorems, they yield simple expressions that can be used to perform calculations that, otherwise, would be much more difficult and technical. Once you know these theorems, you get to a land of simplicity in which many results can be calculated and played with.

This felt like a nice fit to the summer school, so I decided to pursue it. I started writing my "lecture notes" for the tutoring sessions, which would eventually turn into this book. I also taught a very similar course at the II São Paulo School on Gravitational Physics and again at the VI ICTP-SAIFR Summer School for Young Physicists, which further contributed to the present version. Currently, my goal with this text is to introduce the main ideas behind what are black holes and then the laws of black hole thermodynamics. Using them, we can make some simple calculations to understand how these objects behave, and why black hole thermodynamics is so surprising and interesting.

While I use the terminology "black hole thermodynamics" often, I am not sure I would call everything in here by that. "Black hole physics" is fair, though. A few topics simply involve black holes, but focus on other properties that I could not resist to include. Partially because I was afraid I would run out of material in the sessions and wanted to be ready for that. Still, the spirit of including simple calculations is ubiquitous.

Since the goal is for the reader to work out some calculations, the text is filled with exercises. In my opinion, that is what truly distinguishes this book from most outreach books on black holes. This time, you get to have some of the fun of the calculations as well. While at first this may sound a bit scary, in my experience it feels really nice to be able to get to a result yourself, or to understand how the number follow from the principles. Furthermore, seeing the area theorem in action is very helpful in understanding why we care so much about it!

Given that the text has many exercises, choosing whether to include solutions was far from trivial. The tricky bit about physics as a profession is that you always need to figure out the answers for yourself and try to understand whether they are right without someone else telling you. Hence, I was really tempted not to include the solutions. I ended up including them, because the presentation in this notes is very unusual. In a standard textbook, you can start consulting other sources to understand the ideas better. In research, you check related papers. I am afraid, however, that this book may be a bit one-of-a-kind, and thus it would be slightly too cruel to expect the readers to check general relativity textbooks in order to better understand one or other exercise.

Finish the preface

1 Why Do Things Fall?

Some questions about the Universe are so simple anyone could ask them. Yet, they are so profound we can still learn a great deal about reality by attempting to answer them. Some of my favorite examples are

– how did the universe begin?

– what is time?

– what are things made of?

– can we go back in time?

– why do things fall?

I like to call them "silly big questions." While they are very simple to ask, they are very difficult to answer.

Let us think for a moment about why things fall. In Ancient Greece, the answer had to do with what things are made of. Aristotle, for example, believed everything was made out of five elements. These were earth, water, air, fire, and aether. Aether composed the heavenly bodies, while the other elements composed everything we see and find on Earth. The reason things fall is then their natural motion. Each element has a natural tendency to be somewhere. The natural tendency of earth is to stay close to the center of the Earth. The natural tendency of fire is to rise. Water and air are in between them. By following these rules, things fall. A rock falls because falling is its nature.

The nature of matter also determines how objects move. The four elements that occur on the Earth have the natural tendency to move on straight lines, while aether moves in circles. This is Aristotle's explanation for why the Moon orbits the Earth, while throwing a rock does not yield the same effect.

Let us then see what happens when we throw a rock away from us. At first, the rock moves on a straight line (let us say a nearly horizontal line, for example). This is known as violent, or unnatural, motion, because the rock was forced to move on this path by being thrown. At some point, though, the rock gets “tired,” and returns to its natural motion. The nature of the rock is to go closer to the center of the Earth, so it falls down in a straight line.

This may seem silly at first. One of the first things one studies in a physics class is how falling objects actually draw parabolas, which look nothing like two straight lines in a row. The key to notice, however, is that in a physics class one is ignoring drag, while Aristotle did not do this. In fact, it can be shown that Aristotelian physics is in good match with experiment within the limitations the Greeks had at the time. The main difference between them and Newtonian mechanics is that their description always assumed motion inside a fluid (air).

Illustration of ballistics with quadratic drag

In this sense, Aristotle was not wrong. It is more accurate to say he was not very precise. Aristotelian physics provides a good description of day-to-day physics in the sense it approximates the experimental results, but the errors can be noticed if the experiment is sufficiently precise. In particular, Aristotelian physics will fail in more extreme situations.

In high school, and in early undergraduate physics courses, we learn about Newtonian mechanics. Newtonian mechanics is based around three laws.

Newton’s First Law: objects tend to retain their state of motion except under the influence of a force.

Newton’s Second Law: under the influence of a force F, the acceleration a inputted on an object is given by $F = ma$, where m is the object’s mass.

Newton’s Third Law: if an object acts on another with a force F_A, the other reacts on the first with a force F_B of magnitude $F_B = F_A$ and opposite direction.

These are the basics for Newtonian mechanics. When solving practical problems, we typically need to figure out which force is acting, find the acceleration, and then use it to compute the position as a function of time.

In this new framework, what is the explanation for things falling? According to Newton, things fall because there is a force between any two objects with mass that has the tendency of making them attract. The intensity of the force is

$$F = \frac{GMm}{r^2}. \tag{1.1}$$

Above, M and m are the masses of the two bodies we are considering, and r is the distance between them. G is the constant

$$G \approx 6.674\,30 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}, \tag{1.2}$$

known as Newton’s universal constant of gravitation. Since it is measured experimentally, it has some uncertainty. I am omitting this uncertainty for simplicity and quoting only an approximate value, but the error is there and is important if we want to know how good our description is.

Equation (1.1) works well when describing a lot of things. For instance, it can be used as the basis for discussing the motion of planets! Yet, it is still imprecise in some senses. For example, it completely ignores the shape of the objects we are considering. Objects with very different shapes may actually experience gravity in a slightly different way, and Eq. (1.1) actually holds only for spheres (r is the distance between their centers). It is possible to improve our description to account for this, but the point is that there are always imprecisions in any physical theory. Physics is never perfect.

This can be disturbing. We moved from Aristotelian physics to Newtonian physics, but we are still working with a theory that is wrong! Does this mean science is wrong? Not quite. While all physical theories will always have errors and uncertainties, we have a very good understanding of these errors. When moving from Aristotelian physics to Newtonian physics, we made the errors smaller. It may be true that all theories we ever develop will have errors, but they get smaller each time. If the errors are smaller than what we can measure, this is surely enough for practical applications.

Let us consider an interesting example. Suppose you are holding an apple of mass m at a height h above the ground. What is the gravitational force between the Earth and the apple? The answer is

$$F = \frac{GM_{\oplus}m}{(R_{\oplus}+h)^2}, \tag{1.3}$$

where $M_{\oplus}$ is the Earth's mass and $R_{\oplus}$ is its radius. Notice $R_{\oplus}+h$ is the distance from the apple to the center of the Earth. Let me write this expression in a different way. Write is as

$$F = \frac{GM_{\oplus}m}{{R_{\oplus}}^2}\frac{1}{(1+h/R_{\oplus})^2}. \tag{1.4}$$

This is still the same equation, but it hints at something important: $h/R_{\oplus}$ is usually very small.

Let us say we are holding the apple at about $h \approx 10\,\mathrm{m}$. The radius of the Earth is roughly $R_{\oplus} \approx 6400\,\mathrm{km}$. In this case, we get

$$\frac{h}{R_{\oplus}} \approx \frac{10\,\mathrm{m}}{6400\,\mathrm{km}} \approx 0.000\,002. \tag{1.5}$$

Hence, we have the fraction

$$\frac{1}{(1+h/R_{\oplus})^2} \approx 0.999\,999\,688, \tag{1.6}$$

which is very close to 1. Hence, the height at which we hold the apple almost does not matter! We could write

$$F \approx \frac{GM_{\oplus}m}{{R_{\oplus}}^2} \tag{1.7}$$

to a very good precision! This is why many physics problems assume the weight of an object to be constant. The error is still there, but you need a very precise instrument to see a deviation from theory. Sometimes, this may be relevant. If we want to understand how the Universe behaves at a fundamental level, we want to make these errors always smaller. But if we just want to compute how we should throw an object up to hit a target (say, to score three points in a basketball game), this approximation is more than enough.

Is this wrong? Not really. Physics is an experimental science. The true question one should ask is whether our experiments can measure the small differences between our approximation and the "correct" result. Since every measurement comes with an uncertainty, we can ignore effects that would be hidden in the uncertainty anyway.

The modern point of view is that all known physical theories are approximations. We do not have any "right" theories, only precise theories. And that is okay! While one could say all theories are wrong, what is truly important is that some of them are useful. Aristotle's principles fail in scenarios that are much far from what he originally considered, but in some scenarios we get good results and can use that to understand the world.

What about Newtonian physics? How well does it do? While it is a great framework, it also has its drawbacks. For example, the errors will be too big when things are moving too fast or the gravitational fields are too strong. For this, we need a new framework, known as relativity. Even relativity is not perfect, because it still fails when things are too small. In fact, to this day we still do not know how to describe gravitational fields at very small length scales in a precise way. This is known as the problem of quantum gravity.

Problems

Exercise 1 [Universality of Gravity]**:**
An interesting feature of gravity is that its attraction is universal. Show that the acceleration due to Newtonian gravity is the same for all objects, regardless of their mass. ✠

Solution:
Using Newton's law of gravitation and Newton's second law we find

$$F = \frac{GMm}{r^2}, \tag{1.8a}$$
$$ma = \frac{GMm}{r^2}, \tag{1.8b}$$
$$a = \frac{GM}{r^2}. \tag{1.8c}$$

Hence, the masses of the falling objects cancel out. The acceleration depends only on the position of the object and on the source of the gravitational field. ■

2 Basic Ideas of Relativity

Our best theory of how space, time, and gravity works is the general theory of relativity. General relativity is a very rich subject which to this day is being researched. As such, it can be extremely elusive and complicated, but it is equally fascinating. While a complete description of relativity at high school level is impossible, we can still discuss some of its basic features.

Given that physics is an experimental science, it is always safer to think in terms of experiments and measurements. To get a feeling for what we know is relativity, let us begin by picturing moving things. Motion tends to be an intuitive concept in physics.

Suppose Alice is driving a car and Bob is watching from the sidewalk. There are many physical questions we could ask about this scenario. We can ask what is the velocity of the car, its position, its temperature, and so on. Within physics, it does not make sense to ask, for example, what are the car's feelings at the moment. While one could philosophically inquire whether the car has any feelings, we have no way of measuring it.

Let us thus consider the car's velocity. According to Alice, the car is at rest. According to Bob, the car has a non-vanishing velocity. Their measures of velocity disagree. How is this possible?

The key point to be understood is that the word "velocity," by itself, has no meaning. Even in Newtonian physics, different observers can disagree on what they call "velocity." This is because velocity is defined relative to a given observer, and not in an absolute sense. Hence, we can talk about the velocity of the car relative to Alice, or to Bob, but not about an "absolute velocity." We can say that the distance between Alice and the car is not changing, and we can say that the distance between Bob and the car is changing, but there is no universal reference point that we can use to determine whether the car is moving or not. We can only say whether it is moving relative to something else.

It is important to notice that measurements are not relative. If both Alice and Bob measure the "velocity relative to Bob," they both encounter the same result. If a traffic radar measures the velocity of Alice's car and she ends up getting a ticket, both Alice and Bob will agree she received the ticket. And Alice will agree she was moving relative to the traffic radar. This is true even if the velocity of the car relative to Alice was zero. The reason is because "velocity relative to Alice" and "velocity relative to Bob" are different things, even though they share the term "velocity."

The same is true for many other things. For example, position. As I am writing, my computer screen is half a meter in front of me, but it definitely is not half a meter in front of you. When measuring the position of my computer, we always need to specify relative to what. The reason is that we need to specify how the measurement is made. If I stretch a tape measure from my nose to my computer, I get a result. If you stretch a tape measure from your nose to my computer, you get a different result. We can both agree on each other's results, while still disagreeing on what is the distance to the computer (because each one of calls calls a different thing by the name "distance"). Notice, however, that we both agree on the distance between my nose and my computer—after all, we would both measure it in the same way!.

The core ideas of relativity concern which physical quantities are relative (i.e, must always be stated relative to something) and which are not. The main invariant quantity is the speed of light. All non-accelerating observers will always agree on what is the value of the speed of light. Any non-accelerating observer will measure it to be

$$c = 299\,792\,458\,\mathrm{m\,s^{-1}}. \tag{2.1}$$

Notice this did not need to be the case. As we just argued, Alice and Bob can disagree on what is the speed of a car (because the speed relative to Alice and the speed relative to Bob are different concepts). Nevertheless, it turns out that the notion of speed of light relative to any non-accelerating observer is always the same. For this reason, nowadays we do not measure the speed of light anymore. We conventioned that the meter is to be defined as the distance that light travels in $1/299\,792\,458$ seconds. Hence, the value I gave above is exact, with no errors.

This absoluteness of the speed of light has very interesting consequences. For example, in Newtonian physics the notion of time is absolute. In this case, both Alice and Bob agree that the car's engine has been running for, say twenty minutes. All observers would agree on how long something has took if the world was Newtonian. All clocks would be measuring the same thing—an absolute notion of time. Yet, the world is not Newtonian. And this changes a lot.

In relativistic physics, time is relative. The meaning of this is that we cannot talk about how much time it took for something to happen. Only about how much time it took relative to some observer. This is very similar to how "what is the distance to Níck's computer?" does not have an answer unless you specify how this distance should be measured. Is is the distance relative to me? To you? To the Pope? Each of these cases have a different possibility. Each of these cases is measured in a different way, because we need to put the measuring tape on a different person's nose. In relativity, the same is true of time.

I always like to think about this in terms of watches. Let us consider a thought experiment known as the twin paradox experiment. Say Alice and Bob are twins, for the sake of the story. We assume one of them, Alice, enters a spaceship, travels to a distant star, and comes back to the Earth. Bob, meanwhile, stays still on Earth. How long did Alice's travel took according to each of them?

According to Alice's watch, the travel took a certain amount of time. Let us say five years. According to Bob's watch, it took longer. For example, it may have taken ten years. The clocks can disagree because in relativity there is no such thing as "the" time. There is only time relative to Bob's watch, time relative to Alice's watch, and so on. The very concept of time is relative, much like the notion of distance to my computer screen. It does not make sense to ask how long it took for something to happen without specifying how you will be measuring the duration. For instance, it does not make sense to ask how long it took for something to happen without specifying what trajectory the clock will make during the measurement.

This leads to curious scenarios. The "paradox" in the twin paradox is due to the fact that Alice and Bob are twins, by assumption, but at the end of the journey Bob is (literally) five years older than Alice. This specific number (five years) assumes Alice measured five years on her watch and Bob measured ten years in his, but it is always true that the twin who stays on Earth ends up the oldest at the end. The reason is that clocks that accelerate always run slower (i.e., tick less) than clocks that do not. Since Alice eventually has to turn back to return to Earth, she has to accelerate at some point. Since Bob never accelerates, his clock runs faster.

It should be clear that this is not an engineering feature. The effect does not depend on the watches any more than the distance to my computer depend on which tape measure you use. It depends on how the measurement is made, not which instrument is used. For example, suppose we used "aging" as a measurement apparatus. This is very imprecise, but we can tell whether Alice and Bob are in their twenties or in their sixties, for example. We can see their wrinkles, the color of their hair, and the texture of their skin. Then, in relativity, it is completely possible for them to have the same age at Alice's departure, but for Bob to be significantly older at her return. Bob's body will age faster than Alice's. The reason is that time itself only makes sense relative to something, and thus "time relative to Alice" and "time relative to Bob" are different things.

The relativity of time is not the only interesting consequence of relativity. Another curious result is the expression that mass is, in fact, a form of energy. One can show that an object at rest with mass m has an energy content

$$E = mc^2, \tag{2.2}$$

which is perhaps the most famous formula in physics. This energy can be extracted and transformed into other forms, and as thus it is a notion of energy just as valid as any other one.

The fact that the speed of light is the same for all non-accelerated observers also considerably changes the behavior of gravity. In fact, to understand how gravity works in this new setup, we can no longer treat it as a "force." You see, a key problem in Newton's theory of gravity was explaining how gravity was transmitted between two objects. How does the Moon knows the Earth is there in order to orbit it? How does an apple know in which direction it should fall?

The answer comes in the form of "general relativity." "General" is meant in opposition to "special" relativity, which embodies the consequences of relativity in the absence of gravity. Special relativity is a simplified case that can be used as an approximation when gravity is not important, while general relativity also accounts the behavior of gravity. We will study many properties of general relativity, but now is a good moment to list and explain a few general features.

The first curious aspect is, again, that gravity is no longer a force. Instead, it is the universe enforcing inertia. In Newtonian physics, Newton's first law states that objects have a tendency to retain their state of motion, except under the influence of an external force. This is known as inertia. It manifests, for example, when you are sitting in a car that makes a sharp turn. You feel a pull toward the outer side of the curve. While that may seem like a force, and feel like a force, it is a "fictitious force," or "inertial force." There is nothing pulling you, but it feels like it because your body is trying to keep moving in a straight line (according to the principle of inertia), while the car is forcibly making a curve. The net result is you are pulled outward.

Illustration of inertia on a sharp turn

To reconcile gravity with the fixed and finite speed of light, we need to accept it as a similar effect. The orbit of the Moon around Earth and the fall of an apple toward the ground are inertial. They are merely objects retaining their state of motion along spacetime. The "unnatural" behavior is to stop these orbits and falls. For instance, we feel pushed to ground not because gravity is pulling us down, but rather because the ground is pushing us up. We do not feel the force of gravity toward the center of the Earth, but rather the force of the floor keeping us in place.

How come this be? The strange answer is that spacetime is curved. Motion in curved spaces naturally leads to effects similar to fall. For example, consider two people on Earth. Both of them start walking North on straight lines along a meridian. Even though they are walking on straight lines, they end up meeting each other at the North pole. They naturally moved toward each other, even though they did not make any curves (for example, they always walked in the direction of their own noses).

Illustration of meridians converging at the poles

What curves spacetime? Everything. In general relativity, it is common to call everything that is on spacetime as matter. Alice, Bob, their car, their watches, atoms, even light. And all of these things curve spacetime. As Wheeler [2] famously put it: "Spacetime tells matter how to move; matter tells spacetime how to curve."

Gravity is a cosmic dance between the contents of the Universe. Gravitational phenomena are now intrinsically linked to the structure of spacetime itself. The reason the Moon orbits the Earth is because the presence of the Earth curves spacetime in a certain way, and the motion of the Moon on this curved spacetime is such that it orbits the Earth. An apple falls to the Earth by simply moving on the curved spacetime inertially. When it hits the ground, the floor exerts a force on the apple and stops this motion.

As a final comment for now, it is interesting to consider what happens in some extreme scenarios, when the motion of matter on spacetime is very accelerated. For example, two stars orbiting each other. Their motion constantly alters the curvature of spacetime and, incredibly, end up emitting "gravitational waves." Similarly to how disturbing the surface of a lake leads to waves moving away from the disturbance, the motion of matter can create ripples in spacetime itself that will be transmitted to long distances later on. These waves can carry away energy from the system. If the stars merge into a single object, for example, the remnant will typically have less mass than the original pair of stars, because some of the energy got carried away. One of the most exciting scientific developments of the last decade was the direct detection of these waves on Earth.

3 The Schwarzschild Black Hole

Black holes were originally predicted in the framework of general relativity, the best theory of gravitational phenomena we have to this day. General relativity was originally published by Albert Einstein in 1915, and it was originally thought that the equations governing the general relativistic gravitational field—presently known as the Einstein field equations—were too difficult to be solved. Nevertheless, Schwarzschild [7] found an exact solution in 1916.

Schwarzschild had considered a specific case of the Einstein field equations. He assumed the universe contained a single mass, with mass M, and considered the gravitational field around (but outside) the mass. This mass was taken to be spherically symmetric and to not change with time. These assumptions considerably simplify the equations and yield a relatively simple result.

The full details of how the gravitational field works are intricate. To keep things simple, let us focus on one particular question. What is the acceleration one must exert on a small body to keep it at a constant distance from the mass M? In Newtonian gravity, the gravitational force is given by Eq. (1.1), and hence to keep a small body of mass m still at a distance r from the mass M we need to apply a force with magnitude

$$F_{\mathrm{NG}} = \frac{GMm}{r^2}. \tag{3.1}$$

The two forces then cancel out and the small body can stand at rest at a fixed distance from M.

In the Schwarzschild solution, the force that must be exerted on the small body of mass m to keep it still at a distance r from the mass M is

$$F_{\mathrm{GR}} = \frac{GMm}{r^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-\frac{1}{2}}. \tag{3.2}$$

For large values of r, the term in parentheses is approximately 1 and we recover the Newtonian expression. However, for smaller values of r we get an interesting phenomenon. For

$$r = R_S = \frac{2GM}{c^2} \tag{3.3}$$

we find that F_{GR} is infinite. Hence, we would need an infinite force to keep a small particle hovering at a distance R_S from the center of the mass M. R_S is known as the Schwarzschild radius.

Plot with the forces in the Newtonian and relativistic cases

Since I earlier told you gravity is not a force, it may feel weird for me to talk about F_{GR} as a force. The key observation is that F_{GR} is not the force of gravity, but the force required to counter the gravitational pull. For example, this is not the force you feel toward the Earth (there is no such force in general relativity). This is the force the floor needs to exert on your feet to keep you still on the ground.

Since this "holding still" force becomes infinite at the Schwarzschild radius, this suggests a weird behavior, which is the signature of a black hole: once you fall in, it is impossible to come out. Once something falls at $r < R_S$, then it is gone forever. Not even an infinite force would be able to bring it back to the surface.

Since Schwarzschild's solution only assumes a central mass with spherical symmetry, we could also use it to model the surroundings of a star. For example, if we assume the Sun to be roughly spherical, then the Schwarzschild solution would describe the gravitational field on its surroundings (as long as we ignore the gravitational field due to other bodies). In this case, how large is the Schwarzschild radius? Since the mass of the Sun is $\mathrm{M}_\odot \approx 1.99 \times 10^{30}\,\mathrm{kg}$, the Schwarzschild radius of the Sun is $R_{S,\odot} \approx 2.95\,\mathrm{km}$. However, the radius of the Sun is $\mathrm{R}_\odot \approx 6.96 \times 10^5\,\mathrm{km}$. Hence, the Schwarzschild radius of the Sun is much smaller than the radius of the Sun, meaning the Schwarzschild radius is well inside the Sun. Since the Schwarzschild solution only holds outside the Sun, we never get to the Schwarzschild radius. The "holding still" force never really becomes infinite, because inside the Sun we would need to take other effects into account and our previous results is no longer trustworthy. If we want to get to the Schwarzschild radius and see the necessity for an infinite force, we may need to find an object that is extremely compact. It needs to store a lot of mass in a relatively small region.

The weirdness of the Schwarzschild solution does not end there. When dealing with general relativity, the geometry of spacetime is encoded in an object known as the metric. And the metric obtained by Schwarzschild seemed to yield strange results both at the Schwarzschild radius and at the origin $r = 0$ (while this is not a complete argument, do notice that Eq. (3.2) diverges again at $r = 0$). Namely, the metric achieved infinite values in these points, and hence it seemed like the geometry could be very problematic.

In fact, there is not a problem with the metric having strange behaviors in some points. These strangenesses often occur depending on the choice of coordinates you make. General relativity describes spacetime in a way that is very similar to how cartographers describe the Earth, using the same mathematics behind maps and atlases. When you look at a map of the world, there is always a distortion. For example, the areas of the countries near the poles may be much larger than they actually are, or the shapes of some countries and continents may be distorted. It is in fact impossible to make a flat map of the Earth that does not include any sort of distortion. The same thing can happen in general relativity. Some maps yield issues and may lead to a weird behavior of the metric. Hence, before blaming the metric, we must find out whether the problem lies in the geometry or in the map we are using to study the spacetime.

These problems that can occur in the metric are known as singularities. If the problem is actually in the maps we are using, we say it is a coordinate singularity (because it is due to the coordinates in the map). If the problem is indeed in the metric, we say it is a physical singularity. Many physical singularities involve phenomena such as infinite curvature (i.e., infinite gravitational field), but this is not always the case.

We are thus left with a few questions at hand. Are the singularities in the Schwarzschild spacetime physical or not? Furthermore, is there even any sort of object in the universe that is so compact it is actually possible to reach the Schwarzschild radius?

The singularity at the heart of the Schwarzschild solution ($r = 0$) is physical. One can show that a quantity known as the Kretschmann scalar diverges at $r = 0$. The Kretschmann scalar is a quantity that does not depend on the map you use (similarly to how, for example, the temperature of the Earth at each point does not depend on which map you use), and hence if it diverges for one choice of coordinates it diverges for all. Furthermore, it is a geometrical quantity measuring a few aspects of the curvature of the spacetime, and therefore it has physical meaning. One can say the gravitational field is infinite at $r = 0$. Or, even better, would be infinite: physical singularities are not counted as part of spacetime, because general relativity cannot reach those points and the metric never makes sense there.

The singularity at the Schwarzschild radius is a coordinate singularity. By changing from one system of coordinates to another, the problems with the metric disappear, and this was noticed by explicit constructions of new coordinate systems that did not shown any problem. The force needed to keep an object at a fixed distance $r = R_S$ is still infinite: this is a physical and measurable observable that does not change with coordinate changes. However, the metric is well behaved there. This has a simple physical interpretation. While there is nothing out of the ordinary at $r = R_S$ (in the sense that there are no physical barriers or infinite gravitational field or anything of the sense), it is not possible to move from $r < R_S$ to $r > R_S$, regardless of how powerful your rocket engines are. No force is sufficient to escape the region with $r < R_S$. Since the effects of gravity are the same for all objects, this holds true for everything. In particular, not even light is capable of escaping this region. Nowadays, we say this region is a black hole.

There is still the question of whether there are any black holes in the universe. Is it possible to concentrate enough matter in a region so that it becomes a black hole? In other words, is it possible to concentrate an amount of matter inside its own Schwarzschild radius?

The answer is positive, and the first example was given by Oppenheimer and Snyder [8]. They considered a simple model for a very heavy star and showed that the star can contract so much it eventually crosses its own Schwarzschild radius and becomes a Schwarzschild black hole. Hence, these sorts of objects could exist in physically interesting scenarios, as opposed to being a mere mathematical abstraction. Many modern experimental observations are well-explained in terms of black holes. These observations include, for example, the pictures taken by the Event Horizon Telescope (which depict the surroundings of very massive objects believed to be black holes), and the gravitational waves detected by experiments such as LIGO, Virgo, and KAGRA.

Problems

Exercise 2 [Black Stars]**:**
Before the notion of black hole emerged in general relativity, the idea of "black stars" had already occurred to earlier physicists working in Newtonian gravity. Let us investigate this scenario.

(a) Consider a spherical star with mass M and radius R. The escape velocity v_{esc} is defined as the initial upward velocity a particle should have so that it can get infinitely far away from the star and have zero kinetic energy by the time it reaches infinity. Find an expression for v_{esc} in terms of M and R.

(b) Assume now that light has a finite speed c. What is the radius R_c that a star with mass M must have for the escape velocity to be $v_{\text{esc}} = c$? What happens to the escape velocity if the radius is even smaller?

(c) Supposing light is pulled down by gravity, justify why it would make sense for an observer at infinity to call a star with radius smaller than R_c a black star.

(d) Could a black star be considered a black hole? ✠

Solution:
Within Newtonian gravity, the potential energy V for a particle of mass m at a distance r from a mass M is

$$V = -\frac{GMm}{r}. \tag{3.4}$$

The kinetic energy, meanwhile, is given by

$$K = \frac{mv^2}{2}. \tag{3.5}$$

(a) At infinity, a particle that escaped with escape velocity should have zero velocity. Since the potential energy vanishes at infinite distances, this means the total energy must vanish at infinity. Conservation of energy then dictates that, at the surface,

$$\frac{mv_{\text{esc}}^2}{2} - \frac{GMm}{R} = 0. \tag{3.6}$$

Hence,

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}}. \tag{3.7}$$

(b) By using the result from the previous item, we find

$$R_c = \frac{2GM}{c^2}. \tag{3.8}$$

If the radius was smaller, the escape velocity would be larger than the speed of light.

(c) If $R < R_c$, then light can never escape to infinity, as it is pulled back by the collapsed body. In this sense, it is a black star, because an observer at infinity cannot see it.

(d) No. Objects at the surface of a black star cannot escape to infinity, but they can still leave the surface. ■

Exercise 3 [Density of a Black Hole]:
Consider a black hole of mass M. Let us approximate its volume by the volume of a sphere with the radius given by the Schwarzschild radius. This is not a precise approximation, because spacetime is really curved near a black hole and because the notion of "volume of a black hole" does not really make sense (it is actually infinite). Nevertheless, it should give us a rough idea of what is the volume of an object that almost collapsed to a black hole. Compute the mass density of the object assuming it to be homogeneous. Does the density increase or decrease with the mass? Is it possible for an object about to collapse to a black hole to have approximately the density of air (about $1.2\,\mathrm{kg\,m^{-3}}$)? If so, how massive would it be and how large would it be? ✠

Solution:
The Schwarzschild radius is

$$R_S = \frac{2GM}{c^2}. \tag{3.9}$$

Hence, the "volume of a Schwarzschild black hole" would be

$$V_{\text{BH}} = \frac{4\pi}{3} R_S^3, \tag{3.10a}$$

$$= \frac{32\pi G^3 M^3}{3c^6}. \tag{3.10b}$$

The "density of a Schwarzschild black hole" is thus

$$\rho_{\text{BH}} = \frac{M}{V_{\text{BH}}}, \tag{3.11a}$$

$$= \frac{3c^6}{32\pi G^3 M^2}, \tag{3.11b}$$

$$\approx 1.843 \times 10^{19}\,\mathrm{kg\,m^{-3}} \left(\frac{\mathrm{M}_\odot}{M}\right)^2, \tag{3.11c}$$

where $M_\odot$ is the mass of the Sun. Notice this density decreases with the mass.

To reach the density of air, we would need

$$M \approx 3.92 \times 10^9 \, \mathrm{M}_\odot. \tag{3.12}$$

The Schwarzschild radius is then

$$R_S \approx 1.16 \times 10^{13} \, \mathrm{m} \approx 77.4 \, \mathrm{au}, \tag{3.13}$$

where 1 au is the average distance between the Sun and the Earth. ■

Exercise 4 [Negative Masses and a Naked Singularity]**:**
What happens to the Schwarzschild radius if the mass of the black hole is negative? Since the spacetime ends at the singularity at $r = 0$, and there is no event horizon around it, we call it a naked singularity. This may seem strange, since the singularity is a point with "infinite gravity" and we might expect the formation of a horizon before we get to an infinity. To understand this, show using Newtonian gravity and Newton's second law, that if there was an object with negative mass it would repel other objects (regardless of them having positive or negative mass). Hence, the singularity pushes things away, instead of pulling them closer. Do objects with positive mass attract or repel hypothetical objects with negative mass?
While most physicists do not believe in the existence of negative masses or naked singularities, these ideas can be useful thought experiments for us to gather more insight about how gravity works. ✠

Solution:
The Schwarzschild radius becomes negative for $M < 0$. Negative masses are universally repulsive, positive masses are universally attractive. This is due to the behavior of the sign of m in $F = ma$. The equation is $F = |m|a$ for positive masses, while it is $F = -|m|a$ for negative masses. The acceleration of a negative mass points in the direction opposite to the net force acting on it. ■

4 What is a Black Hole?

At this stage, it is nice to discuss what we define as a black hole and give some more examples.

The modern definition of a black hole is, in loose terms, that a black hole is a region in spacetime from which no light rays can infinity. More precisely, if an observer goes infinitely far away after infinite time, they cannot see the interior of the black hole. Let us understand it.

First, we discuss the observers who "go infinitely far away after infinite time". This means the observers are not bound to a small region in spacetime. For example, they can move to very far away, as opposed to being restricted to the solar system, or the galaxy. The goal of mentioning them is to define what we mean by "outside." These observers are outside everything we would like to call black holes (because observers in black holes cannot go to infinity, and hence are not "outside" the black hole). If there is a region in spacetime whose light rays never reach these "outside" observers, we interpret that the light is trapped inside that region. Hence, gravity is so strong not even light can escape. Therefore, we call that a black hole.

We also need the outside observers to avoid defining black holes in a too loose sense. For instance, when we say that "a black hole is a region in spacetime from which not even light can escape," we are being too loose: using that definition, the entire spacetime is a black hole! Indeed, if a light ray always has to be somewhere in the universe, then it can never escape the universe. Nevertheless, that is not what we mean by a black hole. For us, a black hole has to be a finite region. Therefore, we use the observers that go to infinite as a way of determining who is outside and who is inside a black hole.

Notice that the definition of black hole is not local—you do not define a black hole by considering how it behaves in its vicinity. We defined a black hole by saying that no light ray escapes to infinity ever. This means we need to know the whole history of the universe and everything that took place inside it to be able to tell which regions are black holes, because we need to be sure that not a single observer at infinity saw the place we are interested in. Hence, it is a very subtle concept. This is one of the reasons that the mathematics for describing black holes in theoretical physics is complicated.

Once we know where the black holes are, we find that they have a boundary. This boundary is the separation between what goes on inside the black hole, and the outside world. For example, in the Schwarzschild black hole the boundary was the $r = R_S$ surface. We call this the event horizon of the black hole.

Since the Schwarzschild black hole is spherical, we can now calculate the area of its event horizon. In the Schwarzschild spacetime, the r coordinate is defined so that a sphere with constant r has area $A = 4\pi r^2$, mimicking the formula we have in Euclidean geometry. Other choices of coordinate could not have this property, and the fact that spacetime is curved means we cannot blindly trust Euclidean geometry. However, things work fine in the Schwarzschild case, and we find the area of the event horizon to be

$$A = 4\pi R_S^2 = 4\pi\left(\frac{2GM}{c^2}\right)^2 = \frac{16\pi G^2 M^2}{c^4}. \tag{4.1}$$

The area of a black hole is one of its most important properties, as we shall see. One of its interesting properties is that it is a physical and objective quantity, that does not depend on the choice of coordinates.

5 Charged Black Holes

The Schwarzschild black hole is a vacuum solution. This means it assumes spacetime does not contain any forms of matter. In general relativity, "matter" is anything that is on spacetime. This includes, for example, light. We know, however, that light exists. How do we deal with it?

Light is described by the "electromagnetic field." In physics, a "field" is something that permeates spacetime and often mediates interactions among different objects. As an example, imagine a lake, with a paper boat on it. If one throws a rock at the lake, the water will be disturbed, and these disturbances will propagate until they reach the boat, which will then move as a consequence. While the rock and the boat never really touched each other, the water mediated an interaction between them. Fields work in similar ways.

We say that objects that interact with the electromagnetic field are "charged," and the charge measures how strongly they interact. In other words, the more charged an object is, the more intensely it interacts with the electromagnetic field. Charge can also be positive or negative, and this modifies how exactly the interactions happen.

Light is a disturbance—or, more appropriately, a wave—moving through the electromagnetic field. Charged objects cause disturbances on the electromagnetic field just like the rock wiggles the water, and our eyes can later detect these disturbances just like the boat shakes when the water waves pass through it.

For the simple case of two spherical particles with charges Q and q, the intensity of electromagnetic-mediated interaction between them respects the force

$$F = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r^2}, \tag{5.1}$$

which is very similar to Newton's universal law of gravitation, Eq. (1.1). For example, the constant $1/4\pi\epsilon_0$ plays a role very similar to that of G. ϵ_0 is known as the permittivity of free space (or of vacuum) and the value is approximately

$$\epsilon_0 \approx 8.854\,187\,812\,8 \times 10^{-12}\,\mathrm{F\,m^{-1}}. \tag{5.2}$$

I should mention the similarity between gravity and electromagnetism is superficial. Deep down the electromagnetic field and the gravitational field are very different. Notably, everything interacts with gravity, but not everything interacts with electromagnetism. Furthermore, two objects with positive mass attract each other gravitationally, while two objects with the same charge repel each other electromagnetically.

If there are many charges in spacetime, the complete description of the gravitational field will be very complicated, because the electromagnetic field will be complicated too. But there is a simple case in which we are particularly interested: that in which a black hole itself is charged. Suppose we got a Schwarzschild black hole and threw in a charged particle, such as a proton. This would lead to an electromagnetic field in spacetime, and the electromagnetic field will curve spacetime as well. What is the end result?

This scenario was studied not long after the Schwarzschild solution by, among others, Reissner and Nordström [9, 10]. Hence, the solution for a charged black hole that does not spin is now known as the Reissner–Nordström solution.

If we now assume that the mass M at the center of the spacetime has a charge Q, then the gravitational field will behave differently. More specifically, if we want to keep a small mass m with charge q standing still at a distance r from the center, then we need to apply a force

$$F_{\mathrm{GR}} = \left[\frac{GMm}{r^2} - \frac{GQ^2 m}{4\pi\epsilon_0 c^2 r^3}\right]\left(1 - \frac{2GM}{c^2 r} + \frac{GQ^2}{4\pi\epsilon_0 c^4 r^2}\right)^{-\frac{1}{2}} - \frac{Qq}{4\pi\epsilon_0 r^2}. \tag{5.3}$$

Recall that in Newtonian gravity and using standard electrodynamics the force would be

$$F_{\mathrm{NG}} = \frac{GMm}{r^2} - \frac{Qq}{4\pi\epsilon_0 r^2}. \tag{5.4}$$

Notice this equation combines the Newtonian law of universal gravitation and the Coulomb law of electrostatics. The sign difference occurs because equal masses are attracted to each other, but equal charges are repelled by each other.

We see two interesting changes in Eq. (5.3) when compared to Eq. (5.4). First, there is a correction factor to the gravitational attraction which is very similar to the one we had in Schwarzschild spacetime. This is known as a redshift factor and we will get back to it soon. The second new aspect is that the gravitational attraction does not depend only on the mass of the black hole M, but also on its charge Q. This is due to the fact that the electromagnetic field also curves spacetime and also gravitates, and hence the new term corresponds to the gravitation due to the electromagnetic field. Notice, in particular, that this term does not depend on the test particle's charge q, nor does it depend on the sign of Q.

The redshift factor was what allowed us to identify the Schwarzschild radius last time. It led to an infinite force when we got too close to the black hole. When do we get an infinite force this time? First we have the origin $r = 0$, as before. This is indeed a physical singularity. We also get other singularities by studying the redshift factor. We see the force diverges at the points $r_\pm$ with

$$1 - \frac{2GM}{c^2 r_\pm} + \frac{GQ^2}{4\pi\epsilon_0 c^4 r_\pm^2} = 0. \tag{5.5}$$

If we multiply both sides by $r_\pm^2$ we find a quadratic equation, the solution to which is

$$r_\pm = \frac{GM}{c^2} \pm \sqrt{\frac{G^2M^2}{c^4} - \frac{GQ^2}{4\pi\epsilon_0 c^4}}. \tag{5.6}$$

If $Q = 0$, then the two solutions are equal to the Schwarzschild radius. Otherwise, they are different.

Both $r = r_\pm$ are coordinate singularities and they correspond to different regions in the Reissner–Nordström spacetime. While the inside of a Schwarzschild black hole is very monotonous (the only option you have once you enter it is to crash into the singularity after a finite amount of time), the inside of a Reissner–Nordström black hole is much more complex. You may keep traveling inside the black hole and eventually get out in a "new universe." r_+ and r_- codify these sorts of structures. Nowadays, we do not believe the inside of the Reissner–Nordström black hole to be physical—it is too badly behaved and is likely to change to something less strange once we consider that the real world is not spherically symmetric. Nevertheless, the outer regions are still of interest. The event horizon of the black hole is located at the outermost coordinate singularity, $r = r_+$.

We may consider now what is the area of the Reissner–Nordström black hole. Since it is spherically symmetric, it is given by $A = 4\pi r_+^2$. The result is then

$$A = \frac{4\pi G^2 M^2}{c^4}\left(2 + 2\sqrt{1 - \frac{Q^2}{4\pi\epsilon_0 G M^2}} - \frac{Q^2}{4\pi\epsilon_0 G M^2}\right). \tag{5.7}$$

For $Q = 0$ we recover the result we had in Schwarzschild spacetime.

Problems

Exercise 5 [More Naked Singularities]:
The area of a black hole's event horizon should always be a real number. With this in mind, show that there is a maximum amount of charge that a Reissner–Nordström black hole can have. Black holes that exhaust this bound are said to be extremal. A Reissner–Nordström solution that does not correspond to a black hole is an example of a naked singularity. ✠

Solution:
We can tell from Eq. (5.7) that the area of a Reissner–Nordström black hole is a real number only if

$$1 - \frac{Q^2}{4\pi\epsilon_0 G M^2} \geq 0. \tag{5.8}$$

Rearranging this equation yields

$$GM^2 \geq \frac{Q^2}{4\pi\epsilon_0}. \tag{5.9}$$

■

Exercise 6 [Maximum Charge of an Astrophysical Black Hole]:
In principle, a charged black hole can have as much charge as allowed by the bound you derived in Exercise 5. In practice, however, this is not what typically happens. Consider a black hole with mass M and Q. Neglect any effects due to spin. Suppose the proton has charge $q > 0$ and mass m. Notice that experimental data states that

$$\frac{q^2}{4\pi\epsilon_0} > Gm^2. \tag{5.10}$$

Show that for $Q > 0$ there are values of Q that satisfy the bound obtained in Exercise 5, but even so have so much charge that the black hole repels protons instead of attracting them. How large does Q have to be for this to happen? In this scenario, does the black hole attract or repel electrons? Conclude that a black hole with too much charge tends to attract and repel charged particles in such a way that it eventually loses charge. Neglect the relativistic corrections. ✠

Solution:
The total force the black hole exerts on the proton is (5.4)

$$F_{\mathrm{NG}} = \frac{GMm}{r^2} - \frac{Qq}{4\pi\epsilon_0 r^2}. \tag{5.11}$$

Hence, if

$$\frac{Qq}{4\pi\epsilon_0} \geq GMm, \tag{5.12}$$

then the black hole actually repels the proton instead of attracting it. This means we expect a bound

$$Q \leq \frac{4\pi\epsilon_0 GMm}{q}. \tag{5.13}$$

Hence,

$$\frac{Q^2}{4\pi\epsilon_0} \leq \frac{4\pi\epsilon_0 Gm^2}{q^2} GM^2. \tag{5.14}$$

Eq. (5.10) ensures the bound derived on Exercise 5 is respected, and hence our new bound is tighter.
If we plug in the values of q and m for a proton, then we find the bound

$$Q \leq 154\,\mathrm{C}\left(\frac{M}{\mathrm{M}_\odot}\right). \tag{5.15}$$

That means an excess of 9.6×10^{20} protons over electrons in an object with the mass of the Sun (which has about 2×10^{30} kg). We can also write

$$\frac{Q^2}{4\pi\epsilon_0} \leq 8.1 \times 10^{-37} \cdot GM^2. \tag{5.16}$$

This means that, in practice, the charge of an astrophysical black hole is negligible. For this reason, practical calculations often assume the black hole to be uncharged. ■

Exercise 7 [Overcharging a Black Hole]:
Wald [11] considered a few gedankenexperiments (thought experiments) on how one could try to destroy a black hole. In this problem, we will study one of his arguments. Recall that a Reissner–Nordström black hole has a maximum charge allowed by the bound derived in Exercise 5. Consider an extremal black hole, i.e., a black hole with the maximum possible allowed charge. For simplicity, ignore the role of black hole spin. Suppose then you throw inside the black hole a charged particle with charge q and mass m. What is the minimum amount of charge the particle has to have in order to overcharge the black hole once it falls in? Is it possible to overcharge a black hole by dropping a charged particle around it? What do you expect to happen if we "force" the particle to go in by accelerating it toward the black hole? (Hint: recall that, in relativity, the mass of a black hole is an expression of how much energy it contains.) ✠

Solution:
The particle must satisfy $q^2/(4\pi\epsilon_0) > Gm^2$ to overcharge the black hole (with $qQ > 0$). However, this means the black hole will repel the charge due to the electromagnetic interaction. If one tries to "push the charge in," this will mean doing work on the charge, which means increasing the energy of the charge. If the charge goes in, so much work had to be done on it that the increase in mass on the black hole will be larger than the increase in electric charge, and thus the black hole is not destroyed. ■

6 Spinning Black Holes

The next stationary (i.e., equilibrium) black hole solution was obtained only in 1963 by Kerr [12]. The Kerr solution describes a mass M spinning with angular momentum J, and it is much more complicated than the two previous solutions.

Let us first recall what is angular momentum. Angular momentum is a conserved quantity (meaning the total angular momentum is the same before and after a physical process) that measures how much a certain object is spinning. The faster the object revolves, the larger the angular momentum. Furthermore, the mass distribution of the object also influences on how large the angular momentum will be for a given angular velocity. Shortly, the angular momentum is a quantity that characterizes how an object is rotating and "how much" it is rotating.

In the case of a black hole, the angular momentum J is often called "spin." Angular momentum is often thought as something that can be broken down in the movement of many different pieces. For example, the angular momentum of the Earth around the Sun can be computed by considering how each particle in the Earth is moving. In the case of a black hole, however, it is not possible to make this decomposition—the black hole is a spacetime region, not a physical object. When angular momentum is not decomposable, we often call it spin, or "intrinsic angular momentum" (as opposed to "orbital angular momentum").

Since the Kerr black hole spins, it is not spherically symmetric. Spherical symmetry means all directions are equivalent, but the axis of rotation of the black hole singles out a preferred direction. This breaks spherical symmetry. Instead, the symmetries of the Kerr spacetime are the time-translation symmetry (the black hole keeps spinning at the same angular velocity at all times) and axial symmetry, meaning that the axis of rotation is the only special direction.

Due to this lack of spherical symmetry, the force needed to keep a particle still in the Kerr spacetime is much more difficult, and I will not attempt to write it down. In particular, the force depends on the angle between the center of the spacetime, the axis of rotation, and the position in which the test mass is located. It does not depend only on the distance to the black hole, as before.

In addition to that, general relativity predicts an effect known as frame-dragging, or Lense–Thirring precession [13]. Suppose you are standing still above the North pole of a Kerr black hole, along the axis of rotation. You are being kept above the black hole by a powerful rocket. The frame-dragging effect is the prediction that the rotation of the black hole will also make your rocket start spinning. The black hole is a region of spacetime, and spacetime itself is spinning. As a consequence, the spin is transferred onto nearby objects.

This effect is more intense near the equator of the black hole. In fact, if you are sufficiently close, the effect is so intense it becomes impossible to stand still at a fixed distance without rotating around the black hole. Spacetime is dragged around the black hole, and everything in spacetime is dragged along. This region is known as the ergosphere. It is illustrated on Fig. 1.

Beneath the ergosphere we find the black hole's event horizon. The event horizon is not spherical, due to the rotation of the black hole. For sufficiently small spin J, the black hole is just a deformed sphere. The equator gets enlarged, white the poles are flattened down. The faster the black hole spins, the more prominent the effect, until the geometry of the horizon is so warped it is impossible to visualize it in Euclidean space. The largest spin that still admits visualization in three-dimensional Euclidean space is, up to a sign,

$$J = \frac{\sqrt{3}}{2}\frac{GM^2}{c}. \tag{6.1}$$

At this point the event horizon is flat at the poles. This is depicted in Fig. 2. The area of the event horizon of a Kerr black hole is

$$A = \frac{8\pi G^2 M^2}{c^4}\left(1 + \sqrt{1 - \frac{c^2 J^2}{G^2 M^4}}\right). \tag{6.2}$$

Figure 1: Illustration of the ergosphere of a rotating black hole. It is impossible to faithfully draw the physical setup, since it requires representing the curved three-dimensional space between the event horizon and the ergosphere. Nevertheless, some limited illustrations are possible. The sketches above show the ergosphere extending outside the black hole's event horizon. It is wider at the equator and it touches the black hole's poles. Left: side view from the equator. Right: top view. Figure based on Fig. 12.6 in Ref. 3.

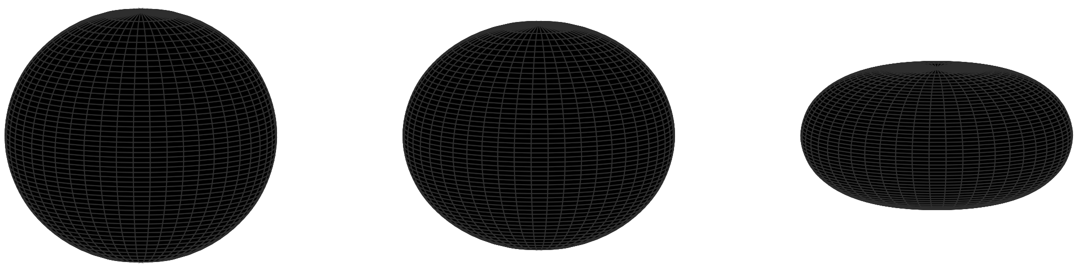

Figure 2: Visual depiction of the event horizon of a Kerr black hole for a few different choices of spin. Left: $J = 0$, corresponding to a Schwarzschild black hole. The event horizon is spherical. Middle: $J = \frac{G}{2c}M^2$. The event horizon is "flattened". Right: $J = \frac{\sqrt{3}G}{2c}M^2$, the maximum spin that still admits a visualization in Euclidean space. The event horizon is flat at the poles.

As with the Reissner–Nordström black hole, we actually get a complicated structure inside the black hole. For example, it is possible to traverse to “different universes,” and the singularity is a ring (as opposed to a point, which was previously the case). In fact, within the event horizon it is even possible to travel back in time (but it is still impossible to leave the event horizon back to where you originally came from). This complicated structure is believed to be an artifact of symmetry, rather than an actual feature present in real black holes.

We are now ready to mix the Reissner–Nordström solution and the Kerr solution into a more general kind of black hole. This more general solution, known as the Kerr–Newman solution, was obtained by Newman and collaborators in 1965 [14], soon after Kerr’s original metric.

The general aspects of the Kerr–Newmann metric are very similar to those of the Kerr metric. The event horizon is not spherical, but rather a “flattened” sphere, and we get Lense–Thirring precession near the black hole. We also have an ergosphere, a rich structure inside the event horizon, and so on.

For us, the most interesting quantity will be the area of the event horizon. It is given by

$$A = \frac{4\pi G^2 M^2}{c^4}\left(2 + 2\sqrt{1 - \frac{Q^2}{4\pi\epsilon_0 G M^2} - \frac{c^2 J^2}{G^2 M^4}} - \frac{Q^2}{4\pi\epsilon_0 G M^2}\right). \tag{6.3}$$

Notice that the special cases $Q = 0$ and $J = 0$ recover the results we had previously in Eqs. (4.1), (5.7) and (6.2).

Problems

Exercise 8 [Some More Naked Singularities]:
The area of a black hole’s event horizon should always be a real number. With this in mind, show that there is a maximum amount for how much charge and spin a black hole can have. Black holes that exhaust this bound are said to be extremal. A Kerr–Newman solution that does not correspond to a black hole is an example of a naked singularity. ✠

Solution:
We can tell from Eq. (6.3) that the area of a Kerr–Newman black hole is a real number only if

$$1 - \frac{Q^2}{4\pi\epsilon_0 G M^2} - \frac{c^2 J^2}{G^2 M^4} \geq 0. \tag{6.4}$$

Rearranging this equation yields

$$G M^2 \geq \frac{Q^2}{4\pi\epsilon_0} + \frac{c^2 J^2}{G M^2}. \tag{6.5}$$

■

7 The No-Hair Theorem

So far, we have discussed a large family of black holes. Something all of them have in common is that they are stationary, which means they do not change in time (at least as far as classical general relativity goes). A Kerr–Newman black hole with mass M, charge Q, and spin J will keep having the very same properties forever.

A key question is then whether there are any other black holes in equilibrium. In other words, consider the following scenario. Through some astrophysical process, a black hole is formed, and it eventually “settles down”, so it is no longer evolving. Is this a Kerr–Newman black hole or could it be different?

The "no-hair theorem" is the epitome of a collection of results in classical general relativity establishing that stationary black holes are uniquely characterized by their mass, charge, and spin. In other words, all stationary black holes are Kerr–Newman black holes. The term "no-hair theorem" was coined by John A. Wheeler in reference to the idea that "hair" makes it easier to distinguish people in a crowded room. Since black holes are completely characterized by three parameters, it is difficult to distinguish them from each other. Nowadays it is common for people to say that "black holes have no hair" in reference to the no-hair theorem.

The present form of the theorem is not as general as we would like it to be. It assumes some very strong mathematical hypotheses on the behavior of the spacetime outside the black hole. For this reason, some authors may prefer to call it the "no-hair conjecture". In spite of this, it is believed a stronger version should be valid, and we just have not been able to prove it yet.

The physical conditions for the no-hair theorem are the following ones.

(a) We assume a single isolated black hole, meaning we are ignoring all other content in the universe.

(b) The black hole is stationary, meaning it is no longer evolving in time.

(c) The only available matter outside the black hole is the electromagnetic field. This means we are ignoring other things that could roam in the spacetime (such as stars and planets, which are not of interest), but most importantly we are ignoring the effect of other forces of nature such as the strong and weak nuclear forces.

Under these conditions, the outside of the black hole (the only region that is accessible for us without falling into the black hole) is the same as for some Kerr–Newman black hole. The inside of a Kerr–Newman black hole is not much trustworthy, so this is not a problem.

For us, this means we can get a lot of insight into black holes by studying the Kerr–Newman solution.

8 The Penrose Process

> I want to consider the question of whether it is possible to extract energy out of a "black hole". One might imagine that, since the matter which has fallen through has been lost for ever, so also is its energy content irrevocably trapped. However, it is not totally clear to me that this need be the case.
>
> Roger Penrose [15]

Now that we know the main examples of black holes, we can start discussing some general properties they must satisfy. The main result we want to consider is Hawking's Area Theorem [4]. We begin with a gedankenexperiment to extract energy from a rotating black hole. At first glance, the idea of extracting energy out of a black hole may seem silly. If by definition a black hole is a region of spacetime from which nothing can come out, how could energy extraction be possible?

Penrose [15] first exemplified an idea proposed to him by Charles Misner. When two black holes coalesce, they typically emit gravitational waves in the process. These gravitational waves are often pictorially described as "ripples through the fabric of space and time", and what really matters for us is that they can carry energy. Hence, merging black holes typically emit a little bit of energy in the merging process. Let us say we have at first two black holes of mass M and that the merging process emits $2kMc^2$ of energy. Then the final black hole has total mass $2M(1-k)$ (we are using the mass-energy equivalence $E = Mc^2$). If we had four black holes of mass M, we would be able to do the process twice and get a final black hole of mass $2^2M(1-k)^2$. If we start with 2^n black holes, doing the process n times means the final black hole has total mass $2^nM(1-k)^n$. No matter how

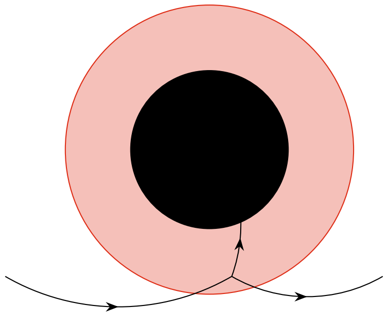

Figure 3: Illustration of the Penrose process from a top view of a Kerr–Newman black hole. The dark circle represents the interior of the black hole (beneath the event horizon), while the shaded orange region represents the ergosphere. A particle is thrown toward the black hole and it splits in two pieces while in the ergosphere. One of the pieces falls down the event horizon, while the other manages to escape the black hole with more energy than the original particle had. Figure based on Fig. 12.7 in Ref. 3.

small k is, if we make n large enough then $(1-k)^n$ will be a very small number, meaning a lot of the original mass inside the black holes was emitted in the form of gravitational waves.

There is, however, a more interesting idea [15, 16]. Suppose we throw in an object at the direction of a Kerr–Newman black hole. Suppose further that, after this object enters the ergosphere, it splits in two pieces: one of which falls down the event horizon, while the other one exits the ergosphere. Then what happens? It turns out that, due to the properties of the ergosphere, this splitting procedure can be arranged so that (to an observer far away from the black hole) the piece that falls down the hole has negative energy, meaning the piece that comes out of the ergosphere has more energy than the one that was originally thrown in. In this way, one can mine energy out of a rotating black hole. This process, known as the Penrose process, is illustrated on Fig. 3.

9 The Irreducible Mass

Now that we know about the Penrose process, how much energy can we extract from, say, a Kerr black hole? This was addressed by Christodoulou [17]. The decrease in black hole mass caused by the Penrose process is accompanied by a decrease in the black hole spin. In this sense, the Penrose process extracts rotational energy from the black hole. Hence, if the black hole eventually becomes a spinless black hole, no more energy can be extracted. By studying how much energy one can extract by removing a given amount of spin from a black hole, Christodoulou showed that the black hole mass can be written in the form

$$M^2 = M_{\text{ir.}}^2 + \frac{c^2 J^2}{4G^2 M_{\text{ir.}}^2}, \tag{9.1}$$

where M is the black hole's mass, J its spin, and $M_{\text{ir.}}$ is its "irreducible mass", meaning the mass of the Schwarzschild black hole obtained after all rotational energy has been extracted from the black hole. Hence, the Penrose process can never make the mass of a black hole smaller than $M_{\text{ir.}}$. This expression assumes the energy has been extracted with maximum efficiency.

It is also possible to factor in electric charge. For a Kerr–Newman black hole, the expression relating the mass, charge, spin, and irreducible mass is [18]

$$M^2 = \left(M_{\text{ir.}} + \frac{Q^2}{16\pi\epsilon_0 G M_{\text{ir.}}}\right)^2 + \frac{c^2 J^2}{4G^2 M_{\text{ir.}}^2}. \tag{9.2}$$

10 The Area Theorem

Let us now grab Eq. (9.2) and solve for M_{ir}. We find, after some algebraic manipulation,

$$M_{\text{ir.}}^2 = \frac{M^2}{4}\left(2 + 2\sqrt{1 - \frac{Q^2}{4\pi\epsilon_0 GM^2} - \frac{c^2J^2}{G^2M^4} - \frac{Q^2}{4\pi\epsilon_0 GM^2}}\right), \tag{10.1}$$

which is more interesting when expressed as

$$\frac{16\pi G^2 M_{\text{ir.}}^2}{c^4} = A = \frac{4\pi G^2 M^2}{c^4}\left(2 + 2\sqrt{1 - \frac{Q^2}{4\pi\epsilon_0 GM^2} - \frac{c^2J^2}{G^2M^4} - \frac{Q^2}{4\pi\epsilon_0 GM^2}}\right). \tag{10.2}$$

By comparing this equation to Eqs. (4.1) and (6.3), we see that the irreducible mass is actually the mass of a Schwarzschild black hole with the same area as the original Kerr–Newman black hole. The irreducibility of the irreducible mass can thus be stated as saying that the area of a black hole may never decrease.

This was originally noticed by Hawking [4], who used more technical methods to establish the result. Due to the generality of the methods employed by Hawking, his result actually holds for many black holes (and then the total area of black holes never decreases) and does not need to be restricted to Kerr–Newman black holes, which means dynamical black holes are also allowed.

Problems

Exercise 9 [Energy Extraction from a Kerr Black Hole]:
Show that the maximum efficiency of energy extraction obtainable through the Penrose process for a Kerr black hole is

$$\frac{M - M_{\text{ir.}}}{M} = \frac{2-\sqrt{2}}{2} \approx 29.3\,\%. \tag{10.3}$$

Hint: the efficiency depends on the spin J. What is the value of J which maximizes the efficiency and what is the value which minimizes it? ✠

Solution:
Since the Penrose process extracts rotational energy, the more rotational energy there is available, the larger the energy that can be extracted. Hence, we consider an extremal Kerr black hole. If we plug the extremality condition derived on Exercise 8 on Eq. (9.1), we find

$$M^2 = M_{\text{ir.}}^2 + \frac{M^4}{4M_{\text{ir.}}^2}. \tag{10.4}$$

This is a biquadratic equation. Solving for M yields

$$M = \sqrt{2}M_{\text{ir.}}. \tag{10.5}$$

Hence, the efficiency is

$$\frac{M - M_{\text{ir.}}}{M} = \frac{\sqrt{2}M_{\text{ir.}} - M_{\text{ir.}}}{\sqrt{2}M_{\text{ir.}}}, \tag{10.6a}$$

$$= \frac{\sqrt{2}-1}{\sqrt{2}}, \tag{10.6b}$$

$$= \frac{2-\sqrt{2}}{2}, \tag{10.6c}$$

as expected. ■

Exercise 10 [Energy Extraction from a Reissner–Nordström Black Hole]**:**
Show that the maximum efficiency of energy extraction for a Reissner–Nordström black hole is

$$\frac{M - M_{\text{ir.}}}{M} = \frac{1}{2} = 50\,\%. \tag{10.7}$$

The reason one can mine energy out of a Reissner–Nordström black hole is that the energy of a particle of charge q and mass m staying at a radius r is given by

$$E = mc^2\left(1 - \frac{2GM}{c^2 r} + \frac{GQ^2}{4\pi\epsilon_0 c^4 r^2}\right)^{\frac{1}{2}} + \frac{Qq}{4\pi\epsilon_0 r}, \tag{10.8}$$

which means that if $Qq < 0$ (the charges have opposite signs) then the energy becomes negative for sufficiently small $r > r_+$. ✠

Solution:
As in the Kerr black hole, we start with an extremal black hole. Using this on Eq. (9.2) yields

$$M^2 = \left(M_{\text{ir.}} + \frac{M^2}{4M_{\text{ir.}}}\right)^2, \tag{10.9}$$

and solving for M yields

$$M = 2M_{\text{ir.}}. \tag{10.10}$$

Hence, the efficiency is

$$\frac{M - M_{\text{ir.}}}{M} = \frac{2M_{\text{ir.}} - M_{\text{ir.}}}{2M_{\text{ir.}}}, \tag{10.11a}$$

$$= \frac{1}{2}, \tag{10.11b}$$

as expected. ■

Exercise 11 [Gravitational Radiation from Colliding Black Holes: Schwarzschild]**:**
Hawking [4, 5] originally used the area theorem to estimate an upper bound on the gravitational radiation emitted on the collision of two Kerr black holes—the same process Misner had earlier suggested could be used to mine energy from black holes. Let us tackle this problem by starting from the simplest scenario: consider two Schwarzschild black holes of masses M_1 and M_2 (assumed to be very far away and "at rest" so the Schwarzschild solution applies) that merge into a single black hole of mass M. Estimate an upper bound on the energy that can be emitted in the form of gravitational waves. Hint: recall that, in relativity, mass is a form of energy with $E = Mc^2$. ✠

Solution:
Conservation of energy tells us that

$$M_1 + M_2 = M + \Delta E c^{-2}, \tag{10.12}$$

where ΔE is the energy emitted in the merger process. If the area of each Schwarzschild black hole is A_1, A_2, and A, then we see that

$$\Delta E c^{-2} = M_1 + M_2 - M, \tag{10.13a}$$

$$= M_1 + M_2 - \frac{c^2}{G}\sqrt{\frac{A}{16\pi}}, \tag{10.13b}$$

$$\leq M_1 + M_2 - \frac{c^2}{G}\sqrt{\frac{A_1 + A_2}{16\pi}}, \tag{10.13c}$$

$$= M_1 + M_2 - \sqrt{M_1^2 + M_2^2}, \tag{10.13d}$$

where we used the area theorem in the form $A \geq A_1 + A_2$ jointly with the fact that $x \mapsto -\sqrt{x}$ is a decreasing function. ■

Exercise 12 [Gravitational Radiation from Colliding Black Holes: Reissner–Nordström]:
Let us build up on Exercise 11 by introducing charge into the problem. Consider two Reissner–Nordström black holes of masses M_1 and M_2 and charges Q_1 and Q_2 (assumed to be very far away and "at rest" so the Reissner–Nordström solution applies) that merge into a single black hole of mass M and charge $Q = Q_1 + Q_2$. Estimate an upper bound on the energy that can be emitted in the form of gravitational waves. Is the emitted energy larger or smaller when the charges have the same sign? Why is that so? Hint: for $x \geq k^2$, the function

$$f(x) = \sqrt{x} + \frac{k^2}{\sqrt{x}} \tag{10.14}$$

is non-decreasing[a]. ✠

[a] You can check this by making a graph for many values of k, but I proved it using calculus.

Solution:
As in Exercise 11, we notice (with the aid of Eq. (9.2)) that

$$\begin{aligned}
\Delta E c^{-2} &= M_1 + M_2 - M, && (10.15a)\\
&= M_1 + M_2 - M_{\text{ir.}} - \frac{Q^2}{16\pi\epsilon_0 G M_{\text{ir.}}}, && (10.15b)\\
&= M_1 + M_2 - \frac{c^2}{G}\sqrt{\frac{A}{16\pi}} - \frac{Q^2}{16\pi\epsilon_0 c^2}\sqrt{\frac{16\pi}{A}}, && (10.15c)\\
&= M_1 + M_2 - \frac{c^2}{\sqrt{16\pi}G}\left(\sqrt{A} + \frac{GQ^2}{\epsilon_0 c^4\sqrt{A}}\right). && (10.15d)
\end{aligned}$$

Next, notice that

$$\begin{aligned}
M &= M_{\text{ir.}} + \frac{Q^2}{16\pi\epsilon_0 G M_{\text{ir.}}}, && (10.16a)\\
\frac{M}{M_{\text{ir.}}} &= 1 + \frac{Q^2}{16\pi\epsilon_0 G M_{\text{ir.}}^2}, && (10.16b)\\
\frac{M - M_{\text{ir.}}}{M_{\text{ir.}}} &= \frac{Q^2}{16\pi\epsilon_0 G M_{\text{ir.}}^2}. && (10.16c)
\end{aligned}$$

Using the results of Exercise 10 we know that $2M_{\text{ir.}} \geq M \geq M_{\text{ir.}}$. It follows that

$$0 \leq \frac{M - M_{\text{ir.}}}{M_{\text{ir.}}} \leq 1. \tag{10.17}$$

Hence,

$$0 \leq \frac{Q^2}{16\pi\epsilon_0 G M_{\text{ir.}}^2} \leq 1. \tag{10.18}$$

In terms of area,

$$0 \leq \frac{GQ^2}{\epsilon_0 c^4 A} \leq 1, \tag{10.19}$$

and hence

$$A \geq \frac{GQ^2}{\epsilon_0 c^4} \equiv k^2. \tag{10.20}$$

Bringing everything together, we can use the area theorem to write

$$\begin{aligned}
\Delta E c^{-2} &= M_1 + M_2 - \frac{1}{\sqrt{16\pi}}\left(\sqrt{A} + \frac{Q^2}{\epsilon_0 G\sqrt{A}}\right), && (10.21a)\\
&= M_1 + M_2 - \frac{c^2}{\sqrt{16\pi G}}\left(\sqrt{A_1 + A_2} + \frac{GQ^2}{\epsilon_0 c^4\sqrt{A_1 + A_2}}\right), && (10.21b)\\
&= M_1 + M_2 - \frac{c^2}{G}\sqrt{\frac{A_1 + A_2}{16\pi}} - \frac{(Q_1 + Q_2)^2}{16\pi\epsilon_0 c^2}\sqrt{\frac{16\pi}{A_1 + A_2}}. && (10.21c)
\end{aligned}$$

When the charges have the same sign, the last term is larger, and hence the energy emitted is smaller. This is consistent with the fact that the electromagnetic force does negative work on the black holes, as they repel each other electromagnetically. Analogously, opposite signs decrease the contribution of the last term and increase the energy available to be scattered by the collision. Notice the areas and masses are not affected by the signs of Q_1 and Q_2. ■

Exercise 13 [Gravitational Radiation from Colliding Black Holes: Kerr]**:**
Following on Exercises 11 and 12, we now consider the case of two Kerr black holes. We assume both of them to be rotating about the same axis, for in this case the total spin is conserved, meaning the final spin is $J = J_1 + J_2$. Estimate an upper bound on the energy that can be emitted in the form of gravitational waves. Is the emitted energy larger or smaller when the spins have the same sign? It turns out that, in general relativity, bodies spinning in opposite directions experience an attraction toward each other [6]. Hint: for $x^2 \geq k^2$, the function

$$f(x) = \sqrt{x + \frac{k^2}{x^2}} \tag{10.22}$$

is non-decreasing[a]. ✠

[a] Once again, I proved it with calculus.

Solution:
Following the solution to Exercise 12, we begin by noticing with the aid of Eq. (9.1) that

$$\begin{aligned}
\Delta E c^{-2} &= M_1 + M_2 - M, && (10.23a)\\
&= M_1 + M_2 - \sqrt{M_{\text{ir.}}^2 + \frac{c^2 J^2}{4G^2 M_{\text{ir.}}^2}}, && (10.23b)\\
&= M_1 + M_2 - \sqrt{\frac{c^4 A}{16\pi G^2} + \frac{4\pi J^2}{c^2 A}}, && (10.23c)\\
&= M_1 + M_2 - \frac{c^2}{\sqrt{16\pi G}}\sqrt{A + \frac{64\pi^2 G^2 J^2}{c^6 A}}. && (10.23d)
\end{aligned}$$

Next, we notice that

$$M^2 = M_{\text{ir.}}^2 + \frac{c^2 J^2}{4G^2 M_{\text{ir.}}^2}, \tag{10.24a}$$

$$\frac{M^2}{M_{\text{ir.}}^2} = 1 + \frac{c^2 J^2}{4G^2 M_{\text{ir.}}^4}. \tag{10.24b}$$

Using the result of Exercise 9, we can establish that

$$1 \leq \frac{M^2}{M_{\text{ir.}}^2} \leq 2. \tag{10.25}$$

It will then follow that

$$0 \leq \frac{c^2 J^2}{4G^2 M_{\text{ir.}}^4} = \frac{64\pi^2 G^2 J^2}{c^6 A^2} \leq 1, \tag{10.26}$$

meaning

$$A^2 \geq \frac{64\pi^2 G^2 J^2}{c^6}. \tag{10.27}$$

Bringing everything together, we get

$$\Delta E c^{-2} = M_1 + M_2 - \frac{c^2}{\sqrt{16\pi G}} \sqrt{A + \frac{64\pi^2 G^2 J^2}{c^6 A}}, \tag{10.28a}$$

$$\geq M_1 + M_2 - \frac{c^2}{\sqrt{16\pi G}} \sqrt{A_1 + A_2 + \frac{64\pi^2 G^2 J^2}{c^6 (A_1 + A_2)}}, \tag{10.28b}$$

$$= M_1 + M_2 - \sqrt{\frac{(A_1 + A_2)c^4}{16\pi G^2} + \frac{4\pi G^2 J^2}{c^6 (A_1 + A_2)}}, \tag{10.28c}$$

$$= M_1 + M_2 - \sqrt{\frac{(A_1 + A_2)c^4}{16\pi G^2} + \frac{4\pi G^2 (J_1 + J_2)^2}{c^6 (A_1 + A_2)}}. \tag{10.28d}$$

As expected, we see the energy emitted is smaller when the spins have the same sign, which is due to the repulsion arising between them due to relativistic effects. When the spins are antiparallel (opposite), the black holes are more strongly attracted to each other and the energy emitted is larger. ■

Exercise 14 [No-Bifurcation Theorem for Schwarzschild Black Holes]:
Another key result about black holes is the no-bifurcation theorem: a single black hole cannot split in two or more pieces [5]. Suppose we start with a Schwarzschild black hole of mass M. Prove that, in the absence of extra initial energy, it is impossible to end with two Schwarzschild black holes of any mass[a]. ✠

[a] I am not sure whether the area theorem can be used to prove this in the general case of Kerr–Newman black holes, but it is possible in the Schwarzschild case.

Solution:
Suppose we could end with two Schwarzschild black holes of masses M_1 and M_2. Then we know that energy conservation demands

$$M \geq M_1 + M_2, \tag{10.29}$$

where we are considering some energy may have been emitted (if there was extra energy at the beginning, we could get two black holes at the end by collapsing the extra energy instead of merging it in the black hole). By taking the square of both sides we find

$$M^2 \geq M_1^2 + M_2^2 + 2M_1M_2. \tag{10.30}$$

Next, we use the area theorem to establish that

$$A_1 + A_2 \geq A, \tag{10.31a}$$
$$16\pi M_1^2 + 16\pi M_2^2 \geq 16\pi M^2, \tag{10.31b}$$
$$M_1^2 + M_2^2 \geq M^2. \tag{10.31c}$$

Bringing everything together we find

$$M_1^2 + M_2^2 \geq M^2 \geq M_1^2 + M_2^2 + 2M_1M_2, \tag{10.32}$$

which implies

$$0 \geq 2M_1M_2. \tag{10.33}$$

Since $M_1, M_2 \geq 0$ (assuming no naked singularities), at least one of the masses must vanish. Hence, we can only have one black hole at the end. ■

11 A Bit of Thermodynamics

The area of a black hole only grows with time. However, it is not the only quantity in physics with this property. The notion of "entropy", which is a core concept in thermodynamics, also has this behavior.

Let us suppose we want to describe a system with a huge number of particles. For example, a gas, which in typical conditions can have about 10^{23} particles. It is impractical, and in practice impossible, to describe the behavior of each of these particles individually. We cannot measure their positions and velocities, and even if we could the calculations would be so complex we would not learn anything from them. So how do we deal with them?

The modern approach to these sorts of problems is to use statistical mechanics. This means using probability theory to describe how the system behaves. Instead of saying what is the position and velocity of each particle and compute the evolution from there, we assign a probability to each possible configuration of positions and velocities. Using these probabilities we can study the average behavior of the gas. While we cannot answer what is the trajectory of a given particle in the gas (we do not even know where it was at the beginning!), we can discuss the temperature, pressure, volume, and other properties of the gas.

How can we decide the probability we should give to each configuration? To understand that, let us for a moment trade the problem for a simpler one. Suppose we have a standard deck of cards with no jokers. There are 52 cards. After the deck is perfectly shuffled, what is the probability that the top card in the deck is the ace of spades?

Let us try to give a few ideas. The first option is that the probability is 1 (*i.e.*, 100%). This is absurd, because we know there are 52 cards and any of them could be on the top! The probability would only be 1 if we had already peaked at the card!

Next attempt. Is the probability 0? This also seems absurd. Any card could be at the top, and there is no reason for us to assume the ace of spades is somewhere else unless we peeked at the deck. Hence, the probability should not be zero.

What about 1/2? We know there are 52 cards and their probabilities should add to 1. Hence, if the ace of spaces has 50% chance of being at the top of the deck, this means there are other cards which are much less likely to be there. We would have no way of telling that unless we peeked at the deck!

We could keep playing this game for a long time. The conclusion is that the lesser evil is to pick the probability as being $1/52$. We cannot say that the probability must be $1/52$, but we can argue against all other possibilities because they seem unnatural. Any other choice implies we have extra information about the deck that suggests the ace of spades should be more or less likely than other cards. In the absence of such information, $1/52$ is the best choice of attribution of probability.

The idea we employed is that the attribution of probability should "maximize the disinformation" in the probability distribution. Unless we know something else about the deck, all cards should have the same probability. Meanwhile, all available information should be used to improve on the probability distribution. For example, since we know the number of cards is 52, we assign $1/52$. We are thus using the total number of cards as information to assign the probabilities.

This is how we assign probabilities in statistical mechanics. We maximize the disinformation—known as entropy—while keeping the information we know. For example, what is the most appropriate probability distribution with the property that the average kinetic energy of the particles (*i.e.*, the temperature) has a given value? Different experimental constraints lead to slightly different probabilities.

Notice this probability is not to be understood as the "true" probability, but rather as the best possible. What is the probability that the top card is the ace of spades? At first $1/52$. Now suppose you look at the bottom card and see it is the two of hearts. Now, immediately, the best probability for the ace of spades being the top card is $1/51$, because there are only 51 candidate cards now (we know where the two of hearts is, and it is not at the top). Hence, we abandon our previous attribution of probability as soon as it becomes uninteresting.

Due to the very nature of entropy, it can only grow with time if we do not "peek" at the deck. I like to think about this in terms of "I can forget information about a system, but I can only gain new information it I look at it". As a consequence, undisturbed systems have a never-decreasing entropy. If we have a closed system (*i.e.*, a system that does not receive interference from outside), its entropy can never decrease. Hence, the total entropy of the universe can only grow. This is very similar to how the total area of black holes in the universe can only grow.

The concept of entropy was known way before we understood it in terms of information, but it was difficult to interpret at first. We knew, however, some of its properties, encoded in the laws of thermodynamics.

There are four laws of thermodynamics. They are the following.

Zeroth Law of Thermodynamics: The temperature of a system in equilibrium is constant throughout the system.

First Law of Thermodynamics: Changes to the energy of a thermodynamical system or to other parameters obey

$$\Delta E = T\Delta S - P\Delta V + \cdots, \tag{11.1}$$

where E is the total internal energy, T is the temperature, S is the entropy, P is the pressure, V is the volume, and the dots denote other possible terms (modifications in the charge, angular momentum, etc).

Second Law of Thermodynamics: The entropy of a closed system cannot decrease,

$$\Delta S \geq 0. \tag{11.2}$$

Third Law of Thermodynamics: The entropy of a system at temperature $T = 0\,\mathrm{K}$ is zero.

There are many subtleties with the third law of thermodynamics, but we will write it in this way for simplicity.

Problems

Exercise 15 [Practicing with Maximum Entropy]**:**
Consider a standard deck of 52 cards. What is the probability that the ace of spades is the top card in each of the following scenarios?

(a) The eight of clubs is at the bottom.

(b) The half top of the deck is comprised of only black suited cards.

(c) Every fourth card is a spade, including the top card. ✠

Solution:
All cases amount to identifying how much information we have about the problem and exploiting it.

(a) Since we know where the eight of clubs is, there are only 51 cards that could be at the top. The probability is $1/51$.

(b) Since all the black cards are at the top half, we only need to know which of them is the uppermost. There are 26 black suited cards, so the probability is $1/26$.

(c) There are thirteen positions in which the ace of spades could be, one of them being the top card. Hence, the probability is $1/13$. ■

12 Black Holes and the Second Law

> The law that entropy always increases—the second law of thermodynamics—holds, I think, the supreme position among the laws of Nature. If someone points out to you that your pet theory of the universe is in disagreement with Maxwell's equations[1]—then so much the worse for Maxwell's equations. If it is found to be contradicted by observation—well, these experimentalists do bungle things sometimes. But if your theory is found to be against the second law of thermodynamics I can give you no hope; there is nothing for it but to collapse in deepest humiliation.
>
> Arthur S. Eddington [19]

A curious property of black holes is the fact that they, at first sight, appear to violate the second law of thermodynamics.

Consider some box filled with matter. Our description of this box includes some entropy S_{mat}, which measures the lack of information we have about the box. For instance, maybe the box contains a deck of cards and S_{mat} measures the 52! possible combinations in which the deck is shuffled. We know that, as long as nothing external interferes with the deck, S_{mat} can only grow.

Now let us assume this box of matter is close to a black hole and will eventually fall into it. If we consider the black hole and the box of matter as a system, their total entropy $S_{\text{mat}} + S_{\text{BH}}$ cannot decrease. But what is the entropy of the black hole?

Let us first determine the temperature of a black hole. To do so, we surround it with a gas at temperature T_0. If the black hole has temperature T, it will be in thermal equilibrium with the gas when $T = T_0$. However, black holes can only absorb particles. Hence, it will continuously devour the gas and only achieve equilibrium for $T_0 = 0\,\text{K}$. Hence, the temperature of a black hole seems to be zero. As such, the third law of thermodynamics states the entropy should also be zero.

[1]Maxwell's equations are the fundamental equations of electromagnetism

The curious thing is that this conclusion holds for any black hole. Let us go back to the process of throwing a box of matter in the black hole. The initial entropy is

$$S_{\text{init}} = S_{\text{mat}} + S_{\text{BH}} = S_{\text{mat}}, \tag{12.1}$$

because $S_{\text{BH}} = 0$. The final entropy, however, is the entropy of a single black hole after absorbing the box of matter. Hence, it is

$$S_{\text{fin}} = S_{\text{BH}'} = 0. \tag{12.2}$$

Hence, $S_{\text{init}} > S_{\text{fin}}$! The entropy of the system has decreased, and thus we have violated the second law of thermodynamics! How can this be?

There is also a second argument, due to Geroch [20]. Suppose the box of matter has energy E_∞ as measured at infinity. General relativity tells us that, for a Schwarzschild black hole, the energy measured at a radius r is given by

$$E(r) = E_\infty \sqrt{1 - \frac{2GM}{c^2 r}}. \tag{12.3}$$

Geroch then argued that lowering the box of matter very slowly until $r \approx \frac{2GM}{c^2}$ would make $E(r) \approx 0$, and thus the box could be dropped inside the black hole. The black hole would not grow (because the box has zero energy when it is dropped), but the entropy inside the box would vanish. Thus, the total entropy of the universe would decrease. Bekenstein [21, 22] solved this by drawing inspiration from information theory. Entropy is a measure of disinformation. Furthermore, it can only grow. Black hole area is thus very similar to entropy in the sense that

i. it can only grow (Hawking's area theorem),

ii. it measures the size of the boundary between the outside universe (about which we can know) and the interior of the black hole (which we are forbidden to see). In this sense, it measures disinformation about the black hole.

Bekenstein then proposed that the entropy of a black hole should be taken to be

$$S_{\text{BH}} = \eta \frac{k_B c^3}{G\hbar} A_{\text{BH}}, \tag{12.4}$$

where A_{BH} is the black hole's area. η is an yet unknown proportionality constant (a number without dimensions). The combination of constants following η is meant to give the correct units on both sides of the equation. k_B is known as the Boltzmann constant,

$$k_B = 1.380\,649 \times 10^{-23}\,\mathrm{J\,K^{-1}}. \tag{12.5}$$

It serves as a conversion factor between the notions of energy and temperature and it has units of entropy. $\hbar$, on the other hand, is known as the reduced Planck constant. Planck's constant is

$$h = 6.626\,070\,15 \times 10^{-34}\,\mathrm{J\,s^{-1}}, \tag{12.6}$$

while $\hbar$ is given by $\hbar = h/2\pi$, thus

$$\hbar = 1.054\,571\,817\ldots \times 10^{-34}\,\mathrm{J\,s^{-1}}, \tag{12.7}$$

which is exact up to the approximation given. This is an important constant in quantum mechanics and it has dimensions of angular momentum. For example, the angular momentum of an electron comes in multiples of $\hbar/2$, and in this sense $\hbar$ quantifies how "granular" is the quantum world.

If black hole area is the black hole entropy, then the second law may not be violated at all! When throwing a box of matter inside the black hole, the black hole grows. This growth then means the total entropy does not need to decrease, and the second law is safe as long as it is generalized: entropy never decreases as long as one accounts for both matter and black holes.

Problems

Exercise 16 [Mass Increase in the Geroch Process]**:**
In the main text I did not address how black hole entropy could account for the paradox given by the Geroch process. This was also addressed by Bekenstein [23] and we will consider it in this and the following problems. Suppose, for simplicity, that the box of matter is spherical with radius R as measured at infinity. Show that if the box has energy E as measured at infinity, then the minimum amount of mass that can be added to the black hole when dropping the box is

$$\delta M_{\min} = \frac{ER}{4GM}. \tag{12.8}$$

You will need to use the fact that, due to the curved geometry near the black hole horizon, an object with radius R (as measured at infinity) put next to the event horizon of a Schwarzschild black will have its center located at the radial coordinate

$$r \approx R_S\left(1 + \frac{R^2}{4R_S^2}\right), \tag{12.9}$$

as opposed to the Euclidean value $R_S + R$. You will also need the approximation

$$\frac{1}{1+x} \approx 1 - x. \tag{12.10}$$

✠

Solution:
Since the box has a finite size, it cannot be brought arbitrarily close to the event horizon. The closest it can come is restricted by its radius R. The smallest coordinate r it can have is then

$$r \approx R_S\left(1 + \frac{R^2}{4R_S^2}\right). \tag{12.11}$$

At this coordinate, the energy measured is

$$\begin{aligned}
E(r) &= E\sqrt{1 - \frac{2GM}{c^2 r}}, && (12.12a)\\
&= E\sqrt{1 - \frac{R_S}{r}}, && (12.12b)\\
&\approx E\sqrt{1 - \frac{R_S}{R_S\left(1 + \frac{R^2}{4R_S^2}\right)}}, && (12.12c)\\
&= E\sqrt{1 - \frac{1}{1 + \frac{R^2}{4R_S^2}}}, && (12.12d)\\
&\approx E\sqrt{\frac{R^2}{4R_S^2}}, && (12.12e)\\
&= E\frac{R}{2R_S}, && (12.12f)\\
&= \frac{Ec^2R}{4GM}. && (12.12g)
\end{aligned}$$

Since $E = mc^2$, the mass added to the black hole when the box is dropped is

$$\delta M_{\text{min}} = E(r)c^{-2} = \frac{ER}{4GM}. \tag{12.13}$$

This means the black hole always grows in the Geroch process. ■

Exercise 17 [Geroch Process and the Bekenstein Bound]**:**
Exercise 16 still does not address how the total entropy of the universe does not decrease. What if the black hole grows just a little and the area increase is not enough to compensate for the decrease in entropy due to the box drop? Bekenstein [23] argued that for this not to happen, the total entropy content S of a spherical box with energy E and radius R (E and R measured at infinity) must respect the (now called) Bekenstein bound

$$S \leq \frac{8\pi\eta k_B ER}{\hbar c}. \tag{12.14}$$

Using the result of Exercise 16, prove this. Hint: use the approximation that the increase in the mass of the black hole during the Geroch process is much smaller than the black hole's mass itself. Notice this argument uses gravitational physics to arrive at a conclusion that does not involve gravitational physics in the sense that G disappears from the equation. ✠

Solution:
The area of a Schwarzschild black hole is given by Eq. (4.1). If after the Geroch process the black hole mass increases by δM (which we will later take to be given by the result of Exercise 16), then the area is increased by

$$\delta A = \frac{16\pi G^2}{c^4}(M + \delta M)^2 - \frac{16\pi G^2 M^2}{c^4}, \tag{12.15a}$$

$$= \frac{16\pi G^2}{c^4}(M^2 + 2\,\delta M \cdot M + \delta M^2) - \frac{16\pi G^2 M^2}{c^4}, \tag{12.15b}$$

$$= \frac{16\pi G^2}{c^4}(2\,\delta M \cdot M + \delta M^2), \tag{12.15c}$$

$$\approx \frac{32\pi G^2 M\,\delta M}{c^4}. \tag{12.15d}$$

Using the result of Exercise 16, we see the minimum area increase of the black hole is

$$\delta A_{\text{min}} \approx \frac{8\pi GER}{c^4}. \tag{12.16}$$

Using Eq. (12.4), we find that the minimum increase in the black hole entropy is

$$\delta S_{\text{BH,min}} \approx \frac{8\pi\eta k_B ER}{\hbar c}. \tag{12.17}$$

To ensure the second law holds, we require that the increase in entropy of the black hole is at least as large as the entropy in the box. Thus,

$$\delta S_{\text{BH,min}} \geq S. \tag{12.18}$$

Hence, we finally get to the Bekenstein bound

$$S \leq \frac{8\pi\eta k_B ER}{\hbar c}. \tag{12.19}$$

■

Exercise 18 [From the Bekenstein Bound to the Spherical Entropy Bound]**:**
Consider again the Bekenstein bound derived in Exercise 17. Assuming the box of matter with energy E, radius R, and entropy S has not collapsed to a black hole, find a bound on the entropy S in terms of the area A of the surface of the spherical box. This is known as the spherical entropy bound [24]. Since the entropy is a measure of (dis)information, this result suggests the content of information of a region is bounded by its area, while we would in principle think it should be its volume. This is believed to be a fundamental principle of quantum gravity, known as the holographic principle, and it may be an important clue to unravelling how gravity works at the quantum level. ✠

Solution:
Since the box has not collapsed to a black hole, its radius must be larger than the Schwarzschild radius. Thus,

$$R > \frac{2GE}{c^4}, \tag{12.20}$$

where I used $E = mc^2$ (for m the mass of the box). Rearranging yields

$$E < \frac{c^4 R}{2G}. \tag{12.21}$$

Now we see the Bekenstein bound yields

$$\begin{aligned} S &\leq \frac{8\pi\eta k_B E R}{\hbar c}, &\text{(12.22a)}\\ &< \frac{8\pi\eta k_B R}{\hbar c}\frac{c^4 R}{2G}, &\text{(12.22b)}\\ &= \frac{4\pi\eta k_B c^3 R^2}{G\hbar}, &\text{(12.22c)}\\ &< \frac{\eta k_B c^3 A}{G\hbar}. &\text{(12.22d)} \end{aligned}$$

■

Exercise 19 [Susskind Process]**:**
There are many criticisms about the Geroch process and its use to derive the Bekenstein bound, and there are also many criticisms about the Bekenstein bound itself. Let us consider a different physical process—the Susskind process [24]—and derive the spherical entropy bound considered in Exercise 18 in a different way. Suppose you now have a spherical box of matter of energy content E, radius R, and entropy S. Suppose you throw in extra matter at the box so that the total mass of the system becomes that of a black hole with area $A = 4\pi R^2$. Show the the entropy S must satisfy the spherical entropy bound derived on Exercise 18 to avoid an entropy decrease. ✠

Solution:
We simply need to have $S \leq S_{\text{BH}}$, where S_{BH} is the entropy of the resulting black hole. We thus have

$$\begin{aligned} S &\leq S_{\text{BH}}, &\text{(12.23a)}\\ &= \eta\frac{k_B c^3}{G\hbar}A_{\text{BH}}, &\text{(12.23b)}\\ &= \eta\frac{k_B c^3}{G\hbar}A, &\text{(12.23c)} \end{aligned}$$

where A is the area of the original spherical box. ■

13 Four Laws of Black Hole Mechanics

A considerable motivation for Bekenstein's proposal that black hole area is proportional to the black hole entropy is Hawking's area theorem. Since the area of a black hole can only increase, we see a resemblance with the second law of thermodynamics, so perhaps they are indeed related after all. Nonetheless, it could still be this is just a mathematical coincidence. The laws of thermodynamics follow from the probabilistic description of systems with a lot of particles, while Hawking's area theorem is geometrical theorem. Their proofs are not at all similar to each other. In addition to that, we have already argued that the entropy of a black hole should be zero, if any. Black holes have no temperature, so they should not have entropy either.

For these reasons, Bekenstein's proposal was originally seem as not more than an analogy, not something to be taken serious. Still, an interesting question is whether the other laws of thermodynamics somehow also appear in black hole physics. This led to the laws of black hole mechanics, first obtained by Bardeen, Carter, and Hawking [25].

We begin with the zeroth law: the temperature of a system in equilibrium is the same throughout the system. We thus expect that some property of black holes in equilibrium is constant throughout the black hole. There is such a quantity, and it is known as surface gravity, usually denoted by κ.

The notion of surface gravity concerns asking: what is the force—or, more appropriately, the acceleration—one must have to stand still at the surface of a black hole? Our discussion of the Schwarzschild and Reissner–Nördstrom black holes revolved around these sorts of questions, and we can thus answer them by looking at Eqs. (3.2) and (5.3). Notably, the force (and thus the acceleration) diverges at the horizon. This is because it is impossible to escape a black hole, so from that point onward no amount of force or acceleration will manage to bring you out. While the acceleration at the surface is infinite, there is a caveat: this is the case for the acceleration as measured at the horizon. What if we could measure it somewhere else?

Let us do the following. Tie a box to a very long rope. Now, we can keep the box still at the surface of the black hole by holding it afar with the rope. If the force on the rope is too large, the rope will break, but we will assume it is a perfect rope that does not break (although they don't sell those anymore). Now stretch the rope from the box (near the black hole) until infinity, so you are holding the box from very afar. The force you need to do on the rope at infinity to keep the box still is given, for a Schwarzschild black hole, by

$$
\begin{aligned}
F_\infty &= \sqrt{1 - \frac{2GM}{c^2 r}} F(r), &\quad (13.1a)\\
&= \frac{GMm}{r^2}, &\quad (13.1b)
\end{aligned}
$$

where $F(r)$ is given by Eq. (3.2) and r is the radial coordinate of the box we are holding. As one approaches the event horizon, we see the acceleration we need to input on the rope at infinity is given by

$$
a_\infty = \frac{F_\infty}{m} = \frac{c^4}{4GM}. \tag{13.2}
$$

We will define this as the surface gravity. Hence, for a Schwarzschild black hole,

$$
\kappa = \frac{c^4}{4GM}. \tag{13.3}
$$

For a Kerr–Newman black hole, the expression is more complicated. It is

$$\kappa = \frac{Mc^4\sqrt{1-\frac{Q^2}{4\pi\epsilon_0 GM^2}-\frac{c^2J^2}{G^2M^4}}}{2GM^2\left(1+\sqrt{1-\frac{Q^2}{4\pi\epsilon_0 GM^2}-\frac{c^2J^2}{G^2M^4}}-\frac{Q^2}{8\pi\epsilon_0 GM^2}\right)}. \tag{13.4}$$

Importantly, notice κ is constant on the event horizon. This is the zeroth law of black hole mechanics: the surface gravity of a black hole in equilibrium is constant throughout the event horizon.

The next question is how changes to the parameters of the black hole affect its area, which would correspond to the first law. The answer is

$$\Delta M = \frac{\kappa}{8\pi G}\Delta A + \cdots, \tag{13.5}$$

where the dots denote the contributions due to changes in the electric charge or spin. Notice that we already started with A being analogous to entropy. The zeroth law told us κ is analogous to temperature, and M is literally the energy of the black hole (up to a factor of c^2). Hence, this equation is indeed very similar to Eq. (11.1).

The second law we already know: $\Delta A \geq 0$, which is Hawking's area theorem.

In this way, we see the mechanics of black holes is very similar to ordinary thermodynamics. Nevertheless, at this point this is merely an analogy: black holes have no temperature, right?

Problems

Exercise 20 [Third Law of Thermodynamics and Extremal Black Holes]**:**
Show that the surface gravity of an extremal black hole (see Exercise 8) is zero, but its area is not. How does this compare to the third law of thermodynamics?
This example shows that the third law of black hole mechanics is more subtle than the others. A different formulation of the third law of thermodynamics states that it is impossible for a system to reach $T = 0\,\mathrm{K}$ in finitely many steps. In this case, it could be translated to black hole mechanics by stating that it takes an infinitely long time for a black hole to become extremal. Bardeen, Carter, and Hawking [25] already believed this to hold because, otherwise, we could be at risk of getting a naked singularity if the process could go further. The rigorous statement was eventually given by Israel [26]. ✠

Solution:
The result follows by plugging in Eq. (6.4) (with an equality) at the expressions for the surface gravity and area of a Kerr–Newman black hole, Eqs. (6.3) and (13.4). ■

Exercise 21 [Revisiting the Bekenstein Bound and the Geroch Process]**:**
In Exercise 17, we used an expression relating the change in the area of a Schwarzschild black hole to the change in its mass—Eq. (12.15). Verify this equation to be consistent with the first law of black hole mechanics. ✠

Solution:
Follows from the first law of black hole mechanics and the expression for κ in the case of a Schwarzschild black hole. ■

14 Superradiance

Our argument for black holes to have no temperature is based on the understanding that nothing can leave them. As a consequence, they can only absorb particles, and are unable to maintain equilibrium at any nonzero temperature.

There is a loophole, though: the Penrose process. We have seen before that by carefully throwing something near a rotating or charged black hole, we can extract energy from it. The trick is that we are actually throwing negative energy in the black hole, rather than actually pulling something out of it. Regardless, the net effect is similar to what we would get if we dragged some energy out of the black hole.

While I talked about this for the case of the Penrose process, which means using particles and breaking them down in an appropriate manner, we could have followed a different route. Suppose, instead, we throw in a wave at the black hole. This could be a light beam, for instance. Just as with the Penrose process, there are situations in which we can arrange for the light beam to emerge from the black hole with more energy than it went in. In more intuitive language, the beam arises from the black hole brighter than it went in!

When we are talking about waves instead of particles, this phenomenon is known as superradiance. It is another example that by throwing something at a rotating black hole, you can sometimes extract something from it (in the sense of the beam coming back with more energy). This is very similar to the phenomenon of stimulated emission in atomic physics—by throwing light at an atom, we can often interact with the atom in precisely the way needed for them to emit even more light in return (this is actually the basic functioning method for the laser). In atomic physics, there is also a phenomenon known as spontaneous emission, in which the atom emits light without the need for external stimulation. This suggests that maybe a rotating black hole could somehow emit energy without the need of throwing something in first.

For this reason, it was believed in the early 1970s that rotating and charged black holes could emit energy in some form. When looking from afar, this would look like the black hole was emitting particles. Hence, there is a change some black holes could be hot after all. We would not expect all black holes to emit particles in this sense. For instance, there is no reason to believe a Schwarzschild black hole would emit energy, especially because if the mass of a Schwarzschild black hole decreases, so does its area.

For these reasons, it was surprising when Hawking [27, 28] showed that even Schwarzschild black holes emit particles. Not only that, but they do it as if the particles were at a fixed temperature, which is proportional to the surface gravity κ. This is exactly what the analogy with thermodynamics would suggest.

15 The Dirac Sea

To understand Hawking's result, I like to first discuss a bit of quantum mechanics. I find the idea of the Dirac sea particularly illuminating.

Quantum mechanics is one of the pillars of modern physics, the other being general relativity. While relativity describes the behavior of spacetime, quantum mechanics—more specifically quantum field theory—is the language in which all other content in the universe is described. For us, the description of electrons will be particularly important.

Within quantum mechanics, electrons are described by an equation known as the Dirac equation. While its details are unimportant for us, a curious property Dirac found in the solutions to his equation (also called states) was that there were not only regular electrons, but also strange solutions with negative energy. Now, this is not negative potential energy, but negative kinetic energy. Hence, it is a very strange feature, and typically considered unphysical. For example, a positive-energy electron could be accelerated by any amount, as long as we did the same to a negative-energy solution. The sum of the energies would cancel out, and we would get a runaway solution. That is, a solution which reaches infinite values in finite time. We do not observe these things in nature, and as a consequence negative energies are usually frowned upon. It was thus a challenge to understand what the negative energy solutions in the Dirac equation meant.

To make sense of these solutions, Dirac used something known as Pauli's exclusion principle. Pauli's principle says that two electrons cannot both occupy the same state at the same time. This means two electrons cannot share all the same properties (energy, momentum, spin, and so on). At least one of them must be different.

Dirac noticed this means that if a solution with negative energy is already "filled" (there is already an electron with those properties), then it is impossible for another electron to carry the same properties. Pauli's exclusion principle forbids it. He then imagined all the negative energy solutions were already filled, but the positive energy solutions were not. In this way, electrons could occupy the free positive-energy spaces, but the negative-energy levels would be forbidden because they were already occupied. Since all of them were occupied, those negative-energy electrons would not be able to get an even-more-negative energy, because there would already be electrons below blocking their way.

This is usually referred to as the Dirac sea. The negative-energy electrons are like the water on the sea, filling everything until the bottom (which, in this case, is infinitely below). Above the see, there is freedom for particles to move, and for the regular electrons to propagate.

What if an electron in the negative-energy sea gains energy by some process (for example, by absorbing a ray of light) and its energy becomes positive? Then it moves up to the positive-energy region, like a droplet leaving the ocean, and leaves a hole behind. The hole in the negative-energy sea then behaves like the opposite of an electron. Other electrons can move into the hole, thus making the hole move around. We can interpret the hole as a particle with positive-energy (because it is a sudden region in which energy is not as negative in the middle of the negative-energy sea) and with positive charge. This is then called an antielectron, or a positron. This behavior is illustrated on Fig. 4.

We see thus that the difference between an electron and a positron concerns whether they are associated to a solution with positive or negative energy. Particles are associated to positive-energy solutions, while antiparticles are associated to negative-energy solutions.

Recall, however, one of the discussions we had at the beginning. It does not make sense to ask about the energy of something, but only the energy relative to a given observer. If both Alice and Bob agree on what is positive and what is negative energy, then they will agree on what are particles and what are antiparticles. However, if they happen to disagree on what is the sign of energy in a few cases, they will disagree on what they call particle.

In the universe, we cannot talk about particles. Only particles relative to some observer!

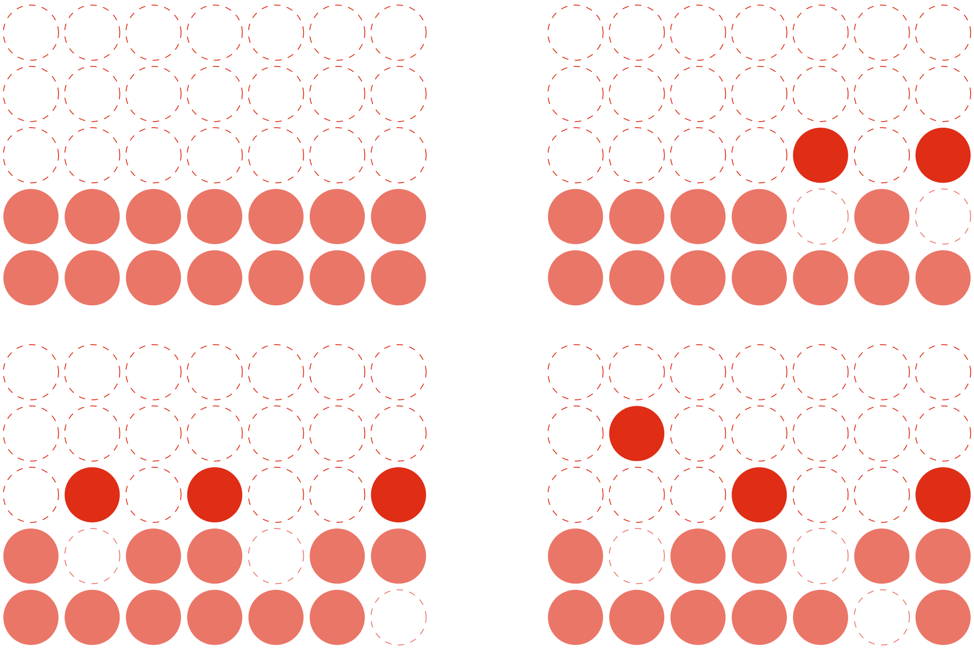

Figure 4: The Dirac sea. Top left: all negative energy states are already filled by electrons. No electrons can get even more negative energy because the lower states are already filled. Top right: some electrons can gain energy and move to the positive-energy region, leaving holes behind them. Bottom left: the positive-energy electrons can move around to different states, and negative-energy electrons can occupy the holes left behind. In practice, the behavior is as if the holes themselves were moving. Bottom right: the positive-energy electrons and positive-energy holes (positrons) behave like opposites of each other and can both move freely.

16 The Hawking Effect

Still writing this one!

17 Advanced Topics

Displacement memory effect and connection to Weinberg soft graviton theorem

Exercise: Braginsky–Thorne gives no memory for mergers

Cosmic censorship conjectures and naked singularities

3D relativity and the BTZ black hole

BHT, Hawking effect

17.1 Interesting Questions to Address

(a) what about white holes?

(b) spaghettification

(c) can black holes move?

(d)

Technical Reading

Since this book is not intended as a technical reference, I omitted most technical references from the heavily influenced by informational perspectives, most notably the discussion by Toffoli [42]. Many interesting developments have happened since the review by Bousso [43] was written, but it still has an excellent discussion of entropy bounds (such as the Bekenstein bound and the spherical entropy bound), their criticisms, and how they relate to ideas in quantum gravity.

Experimental values are typically taken from the Review of Particle Physics [44].

A Gedanken Experiments to Destroy a Black Hole

In this appendix, I describe a seminal paper by Wald [11] which was the topic of the students' final presentation at the original version of this course. This paper was briefly mentioned in Exercise 7, but we now give a more comprehensive discussion.

We expect that, after a black hole is formed—for example through the collapse of a star—it should eventually settle down and stop evolving. When this happens, we expect the no-hair theorem to ensure the resulting black hole is a Kerr–Newman black hole. In other words, after a black hole is formed, we expect it will eventually be described by only its mass, electric charge and angular momentum.

If this is the case, then the results of Exercise 8 tell us that the black hole should respect

$$GM^2 \geq \frac{Q^2}{4\pi\epsilon_0} + \frac{c^2J^2}{GM^2}. \tag{A.1}$$

Otherwise, the area of the event horizon would not be a real number. In practice, this means there is no event horizon, and we do not have a black hole. We would still have some points with infinite curvature in the spacetime, and thus we would have a singularity without having any event horizon. This would be a naked singularity.

Naked singularities are problematic because worldlines—*i.e.*, the "histories" of things moving on spacetime—can fall at them in finite time. For instance, an astronaut could fall in the singularity and their story on spacetime would suddenly end. The astronaut would simply no longer be in the spacetime. The same, however, would be true in reverse: things could in principle come out of the singularity and we would not be able to predict anything about this. Hence, we would lose the ability to predict what happens in the future given the information of what is happening in the universe right now. Since predictions are the core of physics, this is considered very troublesome.

If the singularities are hidden behind black holes, there is no problem. In this case, then the outside of the black hole cannot be affected by the singularity, because nothing can cross the event horizon from the inside to outside. Therefore, we are still able to make predictions in the outside. Hence, physics is much more reliable if all singularities are hidden behind event horizons[2].

Many calculations in black hole physics assume that gravitational collapse (such as the death of a star) cannot lead to a naked singularity. If it did, we would have a lot of difficulty making predictions, and thus the calculations become much more complex to handle. The belief that gravitational collapse cannot lead to naked singularities is known as the "weak cosmic censorship conjecture". As stated by Wald [45],

> The fundamental issue addressed by weak cosmic censorship can be expressed in graphic terms by posing the following question: Could a mad scientist—with arbitrarily large, but finite, resources—destroy the universe?
>
> Robert M. Wald [45]

[2]One notable exception is the Big Bang. While it is not behind an event horizon, there is no problem with having a singularity if it is to the past of everything in the universe. In this case, all things that could come out of it have already come out, and we can still make predictions.

If one could create a naked singularity after gravitational collapse, then its effects could in principle affect distant observers and spoil the good causal behavior of the universe. In other words, knowing the present would not be enough to predict the future. Weak cosmic censorship states we do not have to worry about this.

While the weak cosmic censorship conjecture is a very important statement in general relativity, we still do not have a proof of it. This is what the word "conjecture" implies: we have strong reasons to believe in it, but we still have not been able to give a complete argument to establish that it is indeed true.

With this background in mind, Wald [11] proposed gedankenexperiments to try to violate two conjectures:

i. gravitational collapse cannot lead to naked singularities,

ii. black holes resulting from gravitational collapse eventually settle down to Kerr–Newman black holes.

The goal was not to provide a complete proof that these results are true—this is very difficult—but rather to gain intuition about them. We try to prove them wrong, and if we fail to do so we will believe more strongly in them. Not a proof, but still evidence.

The gedankenexperiments proposed by Wald [11] are then the following. Let us suppose a star collapsed to a black hole and eventually got well-approximated by the Kerr–Newman solution. Then we know (Exercise 8) that the mass M, electric charge Q, and spin J of the black hole must satisfy Eq. (A.1). This then brings an interesting question: can we throw particles into the black hole in a way that we increase Q and J a lot, but increase M so little that Eq. (A.1) eventually breaks down? If so, then we will violate a basic rule of Kerr–Newman black holes. This means that we will either create a naked singularity, or we will end up with a different sort of black hole. Hence, one of the two conjectures will be violated.

The calculations done by Wald use the complete technicalities of the Kerr–Newman spacetime. Hence, I will not follow them in detail, but rather present a simplified version. The purpose of the equations is to understand the ideas better, and for us it will be sufficient to work with a few approximations just to get a feeling for the physics. Of course, the full general relativistic calculations are much more convincing at a research level.

Instead of working with the full Kerr–Newman case, let us consider the charged and spinning cases separately. We begin with the charged case (*i.e.*, a Reissner–Nordström black hole). This is Exercise 7, but I will discuss it again for completeness. We know it satisfies

$$GM^2 \geq \frac{Q^2}{4\pi\epsilon_0}, \tag{A.2}$$

which is just Eq. (A.1) with $J = 0$. We will try to throw in a particle with mass m, energy E and charge q. If the particle starts at rest, then the energy is $E = mc^2$, but if it starts in motion it is larger (we have to add in the kinetic energy). To improve our odds of breaking Eq. (A.2), let us assume we start with an extremal black hole, so the initial state is

$$GM^2 = \frac{Q^2}{4\pi\epsilon_0}. \tag{A.3}$$

If we manage to increase the charge just a bit more than the mass, then we will destroy the black hole. For $Q > 0$ (which we assume for simplicity), we get

$$\sqrt{G}M = \frac{Q}{\sqrt{4\pi\epsilon_0}}. \tag{A.4}$$

Let us first try by dropping a particle from rest. Then $E = mc^2$. If we ignore any gravitational waves, then the final mass of the black hole will be $M + m$, but the final charge will be $Q + q$. If there are gravitational waves, then the final mass of the black hole is smaller than $M+m$ (some energy is lost in the form of gravitational waves), but the charge is still the same. This means our odds of destroying the black hole would be better. Wald did consider this in his calculations, but I will ignore to keep things simple. For the black hole to be destroyed we need

$$G(M+m)^2 < \frac{(Q+q)^2}{4\pi\epsilon_0}. \tag{A.5}$$

It is convenient to rewrite this as

$$\sqrt{G}(M+m) < \frac{Q+q}{\sqrt{4\pi\epsilon_0}}, \tag{A.6}$$

where I assumed $Q+q>0$ for simplicity (the other case needs an extra negative sign).

The force on the test particle due to the black hole will be

$$F \approx \frac{GMm}{r^2} - \frac{1}{4\pi\epsilon_0}\frac{Qq}{r^2}, \tag{A.7}$$

where $F>0$ means the force points toward the black hole. Hence, for the particle to fall toward the black hole we need

$$\frac{GMm}{r^2} > \frac{1}{4\pi\epsilon_0}\frac{Qq}{r^2}, \tag{A.8}$$

and this implies

$$\frac{q}{4\pi\epsilon_0} < \frac{GMm}{Q}. \tag{A.9}$$

Hence, using Eq. (A.4) we find that

$$\begin{aligned}\frac{Q+q}{\sqrt{4\pi\epsilon_0}} &< \frac{Q}{\sqrt{4\pi\epsilon_0}} + \frac{\sqrt{4\pi\epsilon_0}GMm}{Q}, &\text{(A.10a)}\\ &= \sqrt{G}(M+m), &\text{(A.10b)}\end{aligned}$$

which violates Eq. (A.6). Hence, if we drop a particle, it will not be attracted to the black hole if it would destroy it!

We can then try to throw in the particle, instead of merely dropping. In this case, the kinetic energy of the particle will be nonvanishing. Hence, the new mass will not be $M+m$, but $M+Ec^{-2} > M+m$. This turns out to be sufficient to prevent the particle from going in! Hence, if the particle would destroy the black hole, it simply will not enter it!

Wald's original calculation does not consider what happens if we accelerate the particle toward the black hole (he only considers the cases of dropping it, or throwing it in with an initial velocity). Nevertheless, applying a force means doing work on the particle, and hence we expect E to go up and the increase in mass to be larger.

Next let us consider a Kerr black hole. Then Eq. (A.1) implies

$$GM^2 \geq \frac{c^2J^2}{GM^2}, \tag{A.11}$$

which we can rewrite as (I assume $J>0$ for simplicity)

$$\frac{GM^2}{c} \geq J. \tag{A.12}$$

Once again, we consider the extremal case, which is as close to a naked singularity as possible. Then

$$\frac{GM^2}{c} = J. \tag{A.13}$$

Now we try to throw in a particle with mass m and angular momentum j. There are two ways for the particle to have a large value of j: it is either translating around the black hole (like the Earth around the Sun) with a large speed, or it is spinning around itself (like the Earth around itself) at a large angular velocity. Let us consider each case separately.

In the first case (the particle revolves around the black hole), large values of j need a large orbit. It turns out that for j to be large enough for us to get

$$\frac{G(M+m)^2}{c} < J + j. \tag{A.14}$$

This means

$$j > \frac{2GMm}{c} + \frac{Gm^2}{c} > \frac{GMm}{c}. \tag{A.15}$$

Now let us consider a particle spinning with angular momentum j around a mass M. We suppose it is spinning around the equator of the Kerr black hole. In Newtonian gravity (which is simple enough for us to gain intuition), the force on the particle along the radial direction will be

$$F = \frac{GMm}{r^2} - \frac{j^2}{mr^3}, \tag{A.16}$$

where the second term is the centrifugal force. This is the effect we feel in a car making a turn, for example, in which we are pushed toward the door. As before, $F > 0$ means the force points toward the black hole. Notice that if j is very large, then the force will be negative. Using Eq. (A.15), we find

$$\begin{aligned}
F &= \frac{GMm}{r^2} - \frac{j^2}{mr^3}, && \text{(A.17a)} \\
&< \frac{GMm}{r^2} - \frac{G^2M^2m}{c^2r^3}, && \text{(A.17b)} \\
&= \frac{GMm}{r^3}\left(r - \frac{GM}{c^2}\right). && \text{(A.17c)}
\end{aligned}$$

Hence, the force will push the particle outward for $r < GM/c^2$. For an extremal Kerr black hole, this is exactly where the event horizon is located! Hence, the particle will be pushed out of the black hole. Notice the inequality on Eq. (A.15) is actually stronger than the one I used, so the force becomes negative farther away from the horizon (I also ignored the relativistic effects for simplicity).

In practice, what this means is that if j is too large, then the particle will not be able to come close to the black hole. It needs to have a very large orbit to have large values of j, and thus it misses the black hole. Once again, we cannot throw the particle in!

Now what happens if the particle is spinning? In this way, it could also have a very large j, but perhaps without a large orbit. The first remark is that if the particle is orbiting and spinning, then gravitational waves can carry angular momentum away. We thus focus on the case in which the particle is spinning around the same axis as the black hole and is falling on it from above (not from the equator, for example). What happens then?

The trick is that once again a force shows up to prevent the particle from falling in. While this does not happen in Newtonian gravity, in relativity we find that bodies spinning along the same direction repel each other [6]. This repulsion is just enough to prevent the particle from falling in.

These thought experiments do not prove the conjectures we started with are true. However, they show how it can be difficult to violate them. This increases our confidence that the conjectures are true, even though we have not proven them. It appears the universe conspires to prevent us from destroying black holes.

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